Computability in some Fundamental Theorems in Functional Analysis and General Topology

Hiroyasu Kamo

Contents

Chapter 1

Introduction

To study mathematical objects, especially those in analysis, from the point of view of computability is called *computable analysis* or *computability in analysis*. It includes applications of computability theory to analysis, and applications of analysis to computability theory.

In this paper, we show effectivization of some fundamental theorems in functional analysis and general topology. In other words, we show, for some theorems, that the theorem obtained from the original theorem by replacing some topological concepts with their computational counterparts holds as well. It is not always possible to effectivize a theorem.

The fundamental theorems examined in this paper are the contraction theorem, Dini's theorem, and existence and uniqueness of Urysohn's universal metric space. The contraction theorem is as follows:

> *If (M, d) is a complete metric space and $f : M \to M$ is a contraction, then there exists a unique fixed point of f.*

We show:

> *Let $(M, d, \mathcal{S})$ be an effectively locally compact metric space. If $f : M \to M$ is a computable contraction, then* $\operatorname{Fix} f$ *is a computable point.*

Dini's theorem is as follows:

> *If a sequence of real-valued continuous functions on a compact space converges pointwise monotonically to a continuous function, then the sequence converges uniformly to the function.*

We show:

Let $(M, d, \mathcal{S})$ be an effectively compact metric space. Let (f_n) be a computable sequence of real-valued functions on M and f a computable real-valued function on M. If f_n converges pointwise monotonically to f as $n \to \infty$, then f_n converges effectively uniformly to f.

Urysohn's universal metric space is a separable metric space U such that

1. for any separable metric space X, there exists an isometric embedding from X into U, and

2. for any $x_0, \ldots, x_{s-1} \in U$ and any $\alpha_0, \ldots, \alpha_{s-1} \in \mathbb{R}$, if

$$(\forall i, j)\ |\alpha_i - \alpha_j| \leq d_U(x_i, x_j) \leq \alpha_i + \alpha_j,$$

   then there exists $y \in U$ such that

$$(\forall i)\ d_U(x_i, y) = \alpha_i.$$

It has already shown that Urysohn's universal metric space exists and is unique up to isometry. We show that Urysohn's universal metric space is a computable metric space and unique up to computable isometry.

1.1 Brief history of computable analysis

The concept of computability was established in 1930s. The first important result on computability is Gödel's First Incompleteness Theorem, which is roughly:

> If a formal system T is effective and consistent and contains natural number theory, then there exists a sentence that is neither provable nor refutable in T.

Here, effectiveness of a formal system is decidability of the set of all formal proofs. Thus, Gödel's First Incompleteness Theorem is considered one of the origins of computability.

Soon after that, Turing has introduced a virtual machine, which currently called a Turing machine, as a model of computation. Many other models of computation, such as Post system, lambda calculus, etc, are discovered and equivalence of these models to each other was proved. Then, Church proposed to define computability by using arbitrarily one of these equivalent models.

This concept of computability is based on discrete domains, such as Σ^*, $\mathbb{N}$, etc. It is natural that one of the advanced topics is to construct a theory on computability on continuous domains such as Σ^ω, $\mathbb{N}^\omega$, $\mathbb{R}$, etc. This is the beginning of computable analysis.

One of the earliest important results in computable analysis is Rice's discovery of the real closed field of all computable real numbers.

Any real number is a limit of some sequence of rational numbers. Considering this fact, Rice defined *computable real numbers* as follows. A real number x is computable if there exists a sequence of rational numbers, (r_n), such that

1. (r_n) can be generated computably, and
2. (r_n) converges to x effectively.

And he proved that all computable real numbers form a real closed field.

The next epoch was Grzegorchik's discovery of computable real functions.

In analysis, we often use conditions of the form

$$(\forall \varepsilon > 0)(\exists \delta > 0)P(\delta, \varepsilon). \tag{1.1}$$

Here, P satisfies that $\delta' < \delta$, $\varepsilon < \varepsilon'$, and $P(\delta, \varepsilon)$ imply $P(\delta', \varepsilon')$. The method using this kind of conditions is often called "ε-δ method". For example, a function $f : I \to \mathbb{R}$ is uniformly continuous if

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x, x' \in I)[|x - x'| < \delta \implies |f(x) - f(x')| < \varepsilon]. \tag{1.2}$$

In practice, we often show not only the existence of δ for any ε, but also how to calculate a suitable δ from ε. For example, to prove that the function $f : [0, 1] \to \mathbb{R}$, $f(x) = x^2$, is uniformly continuous, we show that $\delta = 2\varepsilon$ is sufficient.

Thus a question arises; whether we would obtain a stronger result by replacing "$\forall\varepsilon\exists\delta$" with "a suitable δ is obtained from ε by some calculation" or not.

To answer to this question, we first formulate "to obtain δ from ε by some calculation". The condition (1.1) is equivalent to:

$$(\exists \varphi : \mathbb{N} \to \mathbb{N})(\forall n \in \mathbb{N})P(1/2^{\varphi(n)}, 1/2^n). \tag{1.3}$$

It is natural that restriction of φ to recursive functions is considered a formulation of "to obtain δ from ε by some calculation". I.e.,

$$(\exists \varphi : \mathbb{N} \to \mathbb{N}, \text{recursive})(\forall n \in \mathbb{N})P(1/2^{\varphi(n)}, 1/2^n). \tag{1.4}$$

For example, effective uniform continuity is a straightforward effectivization of uniform continuity. A function $f : I \to \mathbb{R}$ is effectively uniformly continuous if

$$
\begin{aligned}
&(\exists \varphi : \mathbb{N} \to \mathbb{N}, \text{recursive})(\forall n \in \mathbb{N})(\forall x, x' \in I) \\
&\quad [|x - x'| < 1/2^{\varphi(n)} \implies |f(x) - f(x')| < 1/2^n].
\end{aligned}
\tag{1.5}
$$

A function $f : I \to \mathbb{R}$ is computable if

1. f maps any computable real to a computable real, and,

2. f is effectively uniformly computable.

Since then, the concept of computability is generalized to apply to more spaces. Some of them are:

- Banach space with a computability structure (Pour-El & Richards 1983),

⋆ Computable metric space (Weihrauch 1993),

- Fréchet space with a computability structure (Washihara 1995),

- Effectively compact metric space (Mori & Tsujii & Yasugi 1997),

- Effectively σ-compact metric space (Yasugi & Mori & Tsujii 1999),

- Effective uniform space. (Tsujii & Yasugi & Mori 2001).

1.2 Approaches to computable analysis

Currently, three major approaches to computable analysis are been studied. They are called the *axiomatic approach*, the *representation-based approach*, and the *embedding-based approach*. Most people believe that these three approaches are essentially equivalent. The equivalence however has not yet been totally proved. Ad hoc translation of theorems from one approach to another has been performed.

In the axiomatic approach, some additional relation $\mathcal{S}$ is added to a mathematical structure X and axioms which the appended relations satisfy are stated. The additional relation $\mathcal{S}$ is often called *computability structure*. The set of all computable sequence of points is often used. A function

$f : X \to Y$ is defined to be computable if it preserves the computability structure in some sense.

$$X \xrightarrow{f} Y$$

$$\mathcal{S}_X \mapsto\!\!\rightsquigarrow \mathcal{S}_Y$$

In the representation-based approach, first, choose some *symbol spaces.* The set of all finite sequences of symbols with the discrete topology, denoted by Σ^*, the set of all ω-sequences of symbols with the Cantor topology, denoted by Σ^ω, and the Baire space $\mathbb{N}^\omega$ are often used. Computability of functions from a symbol space to another is defined by using Turing machines or similar virtual machinery. For a space X, define a partial surjective function $\rho_X : S \to X$, which acts as a decoder. A function $f : X \to Y$ is defined to be computable if the following diagram commutes with some computable function φ on some suitable symbol spaces:

$$\begin{tikzcd} S \arrow[r, "\varphi"] \arrow[d, "\rho_X"'] & S' \arrow[d, "\rho_Y"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

In the embedding-based approach, first, choose some *domains.* CPO's (complete partially ordered sets) are often used. Computability of functions from a domain to another is defined by using continuity. For a space X, define an injective function $\iota_X : X \to D$, which acts as an encoder. A function $f : X \to Y$ is defined to be computable if the following diagram commutes with some computable function φ on some suitable domains:

$$\begin{tikzcd} D \arrow[r, "\varphi"] & D' \\ X \arrow[r, "f"'] \arrow[u, "\iota_X"] & Y \arrow[u, "\iota_Y"'] \end{tikzcd}$$

Relationship between computable analysis and analysis without computability appears clearly in the axiomatic approach since it is described as traditional analysis with computability added. However, the axiomatic approach is less powerful than the representation-based approach at present.

In this paper, we use mainly the axiomatic approach and the representation-based approach in part.

1.3 Relations to constructive analysis

Constructive analysis is roughly a system obtained from classical analysis by removing all unconstructive methods. Hence constructive analysis is a proper subsystem of classical analysis, i.e., any theorem of constructive analysis is a theorem of classical analysis but the converse does not hold.

Constructive recursive analysis is roughly a system obtained from constructive analysis by adding some axioms that mean "everything is computable" in a sense. Hence constructive recursive analysis contradicts classical analysis, i.e., some theorem of constructive recursive analysis is a false statement of classical analysis.

Although computable analysis is a part of classical analysis but constructive analysis and constructive recursive analysis are not, computable analysis shares some tricks with constructive analysis and with constructive recursive analysis. For example, the following two facts hold.

($*$) There is no effective procedure to decide for any $x \in \mathbb{R}$ whether $x < a$ or $a \leq x$ even if a is a computable real number.

($**$) Suppose $a, b \in \mathbb{R}$ are computable and $a < b$. There exist effective procedures P, P' such that, for any input $x \in \mathbb{R}$, if P returns true, then $x < b$, and if P' returns true, then $a < x$.

This fact corresponds to the following fact on constructive analysis.

($\sharp$) "For any real number a and any real number x, it holds that $x \leq a$ or $a \leq x$" is not provable in constructive analysis.

($\sharp\sharp$) "For any real numbers a, b with $a < b$ and any real number x, it holds that $x < b$ or $a < x$" is provable even in constructive analysis.

However, there are some major differences between computable analysis and constructive analysis.

- Computable analysis is based on classical logic.

  Both constructive analysis and constructive recursive analysis ara based on intuitionistic logic.

- In computable analysis, existence of uncomputable real numbers is provable.

  In constructive analysis, neither existence nor nonexistence of uncomputable real numbers is provable.

  In constructive recursive analysis, nonexistence of uncomputable real numbers is provable.

1.4 Structure of this paper

In Chapter 2, we summarize definitions of computability on the real line and computability on metric spaces by the axiomatic approach. In Chapter 3, we effectivize the contraction theorem. In Chapter 4, we effectivize Dini's theorem. In Chapter 5, we summarize definitions of computability on metric spaces by the representation-based approach. In Chapter 6, we construct a computable structure of Urysohn's universal metric space and investigate its properties.

Chapter 2

Axiomatic approach to computable analysis

In this chapter, we summarize definitions used in the next two chapters. In §2.1, we define computability on the real line. In §2.2, we define computability on metric spaces by the axiomatic approach.

In this chapter and the next two chapters, we follow the terminology and the notation in [20] except for some minor modifications. Especially, we start natural numbers with 0. We refer to [14] for computability of reals and real functions. We refer to [17] for Type 2 computability.

We abbreviate an ω-sequence to a sequence, an ω^2-sequence to a double sequence, an ω^k-sequence to a k-tuple sequence, etc. We often identify a double sequence $(x_{m,n})$ with a sequence (x_n) such that $x_{m,n} = x_{\langle\langle m,n \rangle\rangle}$ where $\langle\langle m,n\rangle\rangle = m + (m+n)(m+n+1)/2$. This identification is applicable to k-tuple sequences by using a standard construction of tupling from a pairing, $\langle\langle n_1 \rangle\rangle = n_1, \quad \langle\langle n_1, \dots, n_k, n_{k+1} \rangle\rangle = \langle\langle \langle\langle n_1, \dots, n_k \rangle\rangle, n_{k+1} \rangle\rangle$.

We denote the corresponding projection functions by π_i^k, i.e., we define $\pi_i^k(\langle\langle n_1, \dots, n_k \rangle\rangle) = n_i$.

If $\varphi : \mathbb{N} \to \mathbb{N}$ is a recursive function, $(x_{\varphi(n)})$ is said to be an effectively selected subsequence of (x_n).

2.1 Computability in real line

In this section, we summarize definitions on computability in the real line.

Definition 1. A sequence of rational numbers, (r_n), is *computable* if there exists recursive functions $\alpha, \beta, \gamma : \mathbb{N} \to \mathbb{N}$ such that

$$(\forall n) \quad r_n = \frac{\alpha(n) - \beta(n)}{\gamma(n)}.$$

Definition 2. A sequence of real numbers, (x_n), is *computable* if there exists a computable double sequence of rational numbers, (r_{nk}), such that

$$(\forall n)(\forall k) \quad |x_n - r_{nk}| \leq \frac{1}{2^k}.$$

The following proposition is a useful fact in verifying computability of some real numbers.

Proposition 1. *For a sequence of real numbers (x_n), the following two conditions are equivalent:*

- *(x_n) is computable.*
- *There exist a computable sequence of rational numbers (r_n) and a recursive function $\varphi : \mathbb{N} \to \mathbb{N}$ such that*

$$(\forall n) \quad |x_n - r_n| \leq \frac{1}{2^{\varphi(n)}}.$$

2.2 Computability in metric spaces

In this section, we summarize definitions on computability in metric spaces.

Let (M, d) be a metric space. We write $B(a, \varepsilon) = \{x \in M \mid d(a, x) < \varepsilon\}$ and $\bar{B}(a, \varepsilon) = \{x \in M \mid d(a, x) \leq \varepsilon\}$. A double sequence $(x_{n,k})$ is said to converge to (x_n) effectively in n and k as $k \to \infty$ if there exists a recursive function $\psi : \mathbb{N}^2 \to \mathbb{N}$ such that, for any $p, n, k \in \mathbb{N}$, it holds that $k \geq \psi(n, p)$ implies $d(x_{n,k}, x_n) \leq 1/2^p$. A double sequence $(x_{n,k})$ is said to converge to (x_n) uniformly in n and exponentially in k as $k \to \infty$ if for any $n, k \in \mathbb{N}$, it holds $d(x_{n,k}, x_n) \leq 1/2^k$. A double sequence $(x_{n,k})$ is said to be an effective Cauchy sequence as $k \to \infty$ if there exists a recursive function $\varphi : \mathbb{N}^2 \to \mathbb{N}$ such that, for any $p, n, k, k' \in \mathbb{N}$, it holds that $\varphi(n, p) \leq k < k'$ implies $d(x_{n,k}, x_{n,k'}) \leq 1/2^p$.

If $(x_{n,k})$ converges to (x_n) uniformly in n and exponentially in k as $k \to \infty$, then $(x_{n,k})$ converges to (x_n) effectively in n and k as $k \to \infty$. If $(x_{n,k})$ converges to (x_n) effectively in n and k as $k \to \infty$, then there exists an effectively selected subsequence $(x'_{n,k})$ of $(x_{n,k})$ such that $(x'_{n,k})$ converges to (x_n) uniformly in n and exponentially in k as $k \to \infty$.

If $(x_{n,k})$ converges effectively, then $(x_{n,k})$ is an effective Cauchy sequence. If an effective Cauchy sequence converges, it converges effectively. However an effective Cauchy sequence does not always converge.

We summerize here the definitions we will use in this paper.

Definition 3 (Computability structure [13, Definition 5], [20, Definition 1.4]). Let (M, d) be a metric space. A set $\mathcal{S}$ of sequences of points on M is a *computability structure* on (M, d) if the following three conditions hold.

1. *(Metric axiom)* If $(x_m), (y_n) \in \mathcal{S}$, then $(d(x_m, y_n))_{m,n}$ forms a computable double sequence of reals.

2. *(Subsequence axiom)* If $(x_n) \in \mathcal{S}$ and (x'_n) is an effectively selected subsequence of (x_n), then $(x'_n) \in \mathcal{S}$.

3. *(Limit axiom)* If $(x_{n,k}) \in \mathcal{S}$, $(x'_n) \in M^\omega$, and $(x_{n,k})$ converges to (x'_n) effectively in n and k as $k \to \infty$, then $(x'_n) \in \mathcal{S}$.

A sequence in $\mathcal{S}$ is said to be a computable sequence. A point $x \in M$ is said to be a computable point if $(x)_n$, the sequence such that all of its elements equal to x, is a computable sequence.

Mori, Tsujii, and Yasugi investigated effective total boundedness, an effective counterpart of total boundedness, and reached to effectively compact metric spaces and effectively σ-compact metric spaces [13][20]. An effectively compact metric space is a complete and effectively totally bounded metric space. An effectively σ-compact metric space is roughly an effective union of countably many effectively compact metric spaces. We introduce an effectively locally compact metric space as an effectively σ-compact metric space with an additional condition.

Definition 4 (Effectively locally compact metric space). A metric space with a computability structure $(M, d, \mathcal{S})$ is an *effectively locally compact metric space* if the following two conditions hold.

1. *(Completeness)* d is a complete metric.

2. There exist a sequence (K_k) of compact subsets and a computable double sequence $(\xi_{k,i}) \in \mathcal{S}$ such that the following three conditions holds.

   (a) There exist computable functions $\kappa : (\mathbb{N}^\mathbb{N})^2 \to \mathbb{N}$ and $\rho : (\mathbb{N}^\mathbb{N})^2 \to \mathbb{N}$ such that: for any $\boldsymbol{k}, \boldsymbol{i} \in \mathbb{N}^\mathbb{N}$, if $(\xi_{\boldsymbol{k}[n],\boldsymbol{i}[n]})$ converges exponentially in n as $n \to \infty$, then $\bar{B}(\lim_{n\to\infty} \xi_{\boldsymbol{k}[n],\boldsymbol{i}[n]}, 1/2^{\rho(\boldsymbol{k},\boldsymbol{i})}) \subseteq K_{\kappa(\boldsymbol{k},\boldsymbol{i})}$.

(b) *(Effective separability)* $\overline{\{\xi_{k,i} \mid i \in \mathbb{N}\}} = K_k$ for any $k \in \mathbb{N}$.

(c) *(Effective σ-total boundedness)* There exists a recursive function $\sigma : \mathbb{N}^2 \to \mathbb{N}$ such that $\bigcup_{i<\sigma(k,p)} B(\xi_{k,i}, 1/2^p) \supseteq K_k$ for any $k \in \mathbb{N}$.

The definition of an effectively σ-compact metric space [20, Definition 4.2] contains the condition $\bigcup_{k\in\mathbb{N}} K_k = M$ instead of the condition 2a in Definition 4, which is the only difference between the two definitions. Since the condition 2a implies $\bigcup_{k\in\mathbb{N}} K_k = M$, an effectively locally compact metric space is an effectively σ-compact metric space.

Yasugi, Mori, and Tsujii have defined a computable function from an effectively σ-compact metric space to $\mathbb{R}$ [20, Definition 4.3]. We adapt their definition to apply to a function from an effectively locally compact metric space to an effectively locally compact metric space.

Definition 5 (Computable function). Let $(M, d, \mathcal{S})$ be an effectively locally compact metric space with (K_k) and $(\xi_{k,i})$. Let $(M', d', \mathcal{S}')$ be a metric space with a computability structure. A function $f : M \to M'$ is *computable* if the following two conditions hold.

1. *(Sequential computability)* For any $(x_n) \in \mathcal{S}$, it holds $(f(x_n)) \in \mathcal{S}'$.

2. *(Effective uniform continuity)* There exists a recursive function $\psi : \mathbb{N}^2 \to \mathbb{N}$ such that for any $k \in \mathbb{N}$ and $x, y \in K_k$, it holds that $d(x, y) \leq 1/2^{\psi(k,p)}$ implies $d'(f(x), f(y)) \leq 1/2^p$.

The Euclidean line $\mathbb{R}$ with computable sequences of reals satisfies Definition 4. The computability of real functions by Definition 5 coincides with the usual computability.

For a q-tuple of effectively locally compact metric spaces $(M_1, d_1, \mathcal{S}_1)$, $\ldots$, $(M_q, d_q, \mathcal{S}_q)$, construct $(M, d, \mathcal{S})$ by:

$$M = M_1 \times \cdots \times M_q,$$
$$d((x_1, \ldots, x_q), (x'_1, \ldots, x'_q)) = \max\{d_1(x_1, x'_1), \ldots, d_q(x_q, x'_q)\},$$
$$\big((x_1^{(n)}, \ldots, x_q^{(n)})\big)_n \in \mathcal{S} \iff (x_1^{(n)})_n \in \mathcal{S}_1 \wedge \cdots \wedge (x_q^{(n)})_n \in \mathcal{S}_q.$$

Then $(M, d, \mathcal{S})$ forms an effectively locally compact metric space, which we call the product space of $(M_1, d_1, \mathcal{S}_1)$, $\ldots$, $(M_q, d_q, \mathcal{S}_q)$. If $(M_1, d_1, \mathcal{S}_1)$, $\ldots$, $(M_q, d_q, \mathcal{S}_q)$ are effectively locally compact metric spaces with $(K_1^{(k)})$, $(\xi_1^{(k,i)})$, ρ_1, κ_1, σ_1, $\ldots$, $(K_q^{(k)})$, $(\xi_q^{(k,i)})$, ρ_q, κ_1, σ_q, respectively, then $(M, d, \mathcal{S})$ is an effectively locally compact metric space with (K_k), $(\xi_{k,i})$, ρ, κ, σ defined by:

$$K_k = K_1^{(\pi_1^q(k))} \times \cdots \times K_q^{(\pi_1^q(k))},$$
$$\xi_{k,i} = (\xi_1^{(\pi_1^q(k),\pi_1^q(i))}, \dots, \xi_q^{(\pi_q^q(k),\pi_q^q(i))}),$$
$$\rho(\varphi) = \max\{\rho_1((\pi_1^q \times \pi_1^q) \circ \varphi), \dots, \rho_q((\pi_q^q \times \pi_q^q) \circ \varphi)\},$$
$$\kappa(\varphi) = \langle\!\langle \kappa_1((\pi_1^q \times \pi_1^q) \circ \varphi), \dots, \kappa_q((\pi_q^q \times \pi_q^q) \circ \varphi)\rangle\!\rangle,$$
$$\sigma(k,p) = \max\{\sigma_1(\pi_1^q(k),p), \dots, \sigma_q(\pi_q^q(k),p)\}$$

where $\pi_1^q, \dots, \pi_q^q$ denote the projections.

Definition 6 (Effectively compact metric spaces). A metric space with a computability structure $(M, d, \mathcal{S})$ is an *effectively compact metric space* if the following two conditions hold.

1. *(Completeness)* d is a complete metric.
2. There exist a sequence $(\xi_i) \in \mathcal{S}$ such that the following two conditions hold.
   (a) *(Effective separability)* $\overline{\{\xi_i \mid i \in \mathbb{N}\}} = M$.
   (b) *(Effective total boundedness)* There exists a recursive function $\sigma : \mathbb{N} \to \mathbb{N}$ such that $\bigcup_{i<\sigma(p)} B(\xi_i, 1/2^p) = M$ for any $k \in \mathbb{N}$.

If $(M, d, \mathcal{S})$ is an effectively compact metric space with (ξ_i), then it is an effectively locally compact metric space with $(M)_k$ and $(\xi_i)_{k,i}$. We consider an effectively compact metric space a special case of effectively locally compact metric spaces.

For a relation $S \subset X_1 \times \cdots \times X_m \times Y_1 \times \cdots \times Y_n$ and an m-tuple $(x_1, \dots, x_m) \in X_1 \times \cdots \times X_m$, the set $\{(y_1, \dots, y_n) \in Y_1 \times \cdots \times Y_n \mid (x_1, \dots, x_m, y_1, \dots, y_n) \in S\}$ is denoted by $S(x_1, \dots, x_m)$.

We fix some standard tuple functions and corresponding projection functions on $\mathbb{N}$. $\langle \text{-}, \dots, \text{-} \rangle$ denotes the n-tuple function. $(\text{-})_k^n$ denotes the corresponding kth projection function. We often identify an n-tuple sequence $(x_{k_1,\dots,k_n})$ with its serialization $(x_{(k)_1^n,\dots,(k)_n^n})_{k\in\mathbb{N}}$.

We use the terminology and the notation on computability of real numbers and of real functions that Pour-El and Richards have used in [14].

The following four definitions are introduced in [20] by Yasugi, Mori, and Tsujii.

Definition 7. Let (M, d) be a metric space. A set $\mathcal{S} \subset M^\omega$ is a *computability structure* on (M, d) if the following three conditions hold.

1. If $(x_n), (y_n) \in \mathcal{S}$, then $(d(x_n, y_{n'}))_{n,n'}$ forms a computable double sequence of real numbers.

2. If $(x_n) \in \mathcal{S}$ and $\sigma : \mathbb{N} \to \mathbb{N}$ is a recursive function, then $(x_{\sigma(n)}) \in \mathcal{S}$.

3. If $(x_{n,k}) \in \mathcal{S}$, $(x'_n) \in M^\omega$, and $(x_{n,k})$ converges to (x'_n) effectively in n and k as $k \to \infty$, then $(x'_n) \in \mathcal{S}$.

An element of $\mathcal{S}$ is called a *computable sequence* in M.

Definition 8. A metric space with a computability structure $(M, d, \mathcal{S})$ is *effectively totally bounded* if there exists a computable sequence $(e_l) \in \mathcal{S}$ and a recursive function $\gamma : \mathbb{N} \to \mathbb{N}$ such that

$$M = \overline{\{e_l \mid l \in \mathbb{N}\}} \quad \text{and} \quad (\forall i \in \mathbb{N})\ M = \bigcup_{l=0}^{\gamma(i)} B(e_l, 1/2^i).$$

$(M, d, \mathcal{S})$ is *effectively compact* if it is effectively totally bounded and d is a complete metric.

Definition 9. Let $(M, d, \mathcal{S})$ be a metric space with a computability structure. A subset $K \subset M$ is an *effectively compact subset* of M if $(K, d|_K, \mathcal{S} \cap K^\omega)$ forms an effectively compact metric space.

In other words, K is an effectively compact subset of $(M, d, \mathcal{S})$ iff it is a compact subset of (M, d) and there exists a computable sequence $(e_l) \in \mathcal{S}$ and a recursive function $\gamma : \mathbb{N} \to \mathbb{N}$ such that

$$K = \overline{\{e_l \mid l \in \mathbb{N}\}} \quad \text{and} \quad (\forall i \in \mathbb{N})\ K \subset \bigcup_{l=0}^{\gamma(i)} B(e_l, 1/2^i).$$

Definition 10. Let $(M, d, \mathcal{S})$ be an effectively compact metric space. A sequence of functions (f_n), $f_n : M \to \mathbb{R}$, is *computable* if the following two conditions hold.

1. (*Sequential computability*) If $(x_k) \in \mathcal{S}$, then $(f_n(x_k))_{n,k}$ forms a computable sequence of real numbers.

2. (*Effective uniform continuity*) There exists a recursive function $\alpha : \mathbb{N}^2 \to \mathbb{N}$ such that for any $n, j \in \mathbb{N}$ and any $x, y \in M$,

$$d(x, y) < \frac{1}{2^{\alpha(n,j)}} \implies |f(x) - f(y)| < \frac{1}{2^j}.$$

A function $f : M \to \mathbb{R}$ is a *computable function* if $(f)_{n\in\mathbb{N}}$, the sequence whose elements are always equal to f, is a computable sequence of functions.

The recursive function α in Definition 10 (2) is called an *effective modulus of continuity* of (f_n).

Chapter 3

Contraction Theorem

In this chapter, we introduce an effective version of the contraction theorem.

Let (X, d) be a metric space. A function $f : X \to X$ is called a *contraction* if there exists a real L with $0 < L < 1$ such that $d(f(x), f(y)) \leq Ld(x, y)$ for any $x, y \in X$. If (X, d) is a complete metric space and $f : X \to X$ is a contraction, then there exists a unique fixed point of f. This theorem is known as the *contraction theorem* and of wide application.

We will show in §3.1 that if the contraction is a computable function on an effectively locally compact metric space, then the fixed point is a computable point on the space.

Many facts on computability can be proved by using the effective contraction theorem. We will give an example in §3.2. The example is the result on computability of self-similar sets by Kamo and Kawamura. Hutchinson [8] proved a theorem on existence of self-similar sets and Hata [6] generalized it. Kamo and Kawamura [10] added the viewpoint of computability to it. We will rewrite Kamo and Kawamura's proof by using the effective contraction theorem.

3.1 Effective Contraction Theorem

We denote by $\mathrm{Fix}\, f$ the unique fixed point of a contraction f.

We use the following three lemmata in this section.

Lemma 1 (Effective completeness [20, Proposition 1.4]). *Let $(M, d, \mathcal{S})$ be an effectively locally compact metric space. If a computable double sequence $(x_{n,k})$ is an effective Cauchy sequence as $k \to \infty$, then there exists a computable sequence (x_n) such that $(x_{n,k})$ converges to (x_n) as $k \to \infty$ effectively in n and k.*

Lemma 2 (Effective density lemma [13, Proposition 1.2]). *Let $(M, d, \mathcal{S})$ be an effectively locally compact metric space with (K_k) and $(\xi_{k,i})$. For a sequence $(x_n) \in M^\omega$, the following three conditions are equivalent.*

1. *(x_n) is a computable sequence.*
2. *There exists a recursive function $\varphi : \mathbb{N}^2 \to \mathbb{N}^2$ such that $(\xi_{\varphi(n,l)})$ converges to (x_n) effectively in n and l as $l \to \infty$.*
3. *There exists a recursive function $\varphi : \mathbb{N}^2 \to \mathbb{N}^2$ such that $(\xi_{\varphi(n,l)})$ converges to (x_n) uniformly in n and exponentially in l as $n \to \infty$.*

Lemma 3 (Iteration). *Let $(M, d, \mathcal{S})$ be an effectively locally compact metric space with (K_k) and $(\xi_{k,i})$. If $a \in M$ is a computable point and $f : M \to M$ is a computable function, then $(f^n(a))_{n\in\mathbb{N}}$ forms a computable sequence of points.*

Proof. Since $d(\xi_{k,i}, \xi_{k',i'})$ forms a computable quadruple sequence of reals, there exists a recursive function $\alpha : \mathbb{N}^5 \to \mathbb{Q}$ such that

$$(\forall k, i, k', i', l \in \mathbb{N}) \quad |d(\xi_{k,i}, \xi_{k',i'}) - \alpha(k, i, k', i', l)| \leq \frac{1}{2^l}$$

Let $\kappa : (\mathbb{N}^2)^{\mathbb{N}} \to \mathbb{N}$ and $\rho : (\mathbb{N}^2)^{\mathbb{N}} \to \mathbb{N}$ be computable functions such that: for any $\varphi : \mathbb{N} \to \mathbb{N}^2$, if $(\xi_{\varphi(n)})$ converges exponentially in n as $n \to \infty$, then $B(\lim_{n\to\infty} \xi_{\varphi(n)}, 1/2^{\rho(\varphi)}) \subseteq K_{\kappa(\varphi)}$.

We use Lemma 2 (*1 ⇒ 3*) twice. Since a is a computable point, there exists a recursive function $\varphi' : \mathbb{N} \to \mathbb{N}^2$ such that $d(\xi_{\varphi'(l)}, a) \leq 1/2^l$ for any $l \in \mathbb{N}$. Since f is sequentially computable, $(f(\xi_{k,i}))$ forms a computable double sequence. Thus there exists a recursive function $\varphi'' : \mathbb{N}^3 \to \mathbb{N}^2$ such that $d(\xi_{\varphi''(k,i,l)}, f(\xi_{k,i})) \leq 1/2^l$ for any $k, i, l \in \mathbb{N}$.

Since f is effectively uniformly continuous, there exists a recursive function $\psi : \mathbb{N}^2 \to \mathbb{N}$ such that for any $k \in \mathbb{N}$ and $x, y \in K_k$, it holds that $d(x, y) \leq 1/2^{\psi(k,p)}$ implies $d'(f(x), f(y)) \leq 1/2^p$.

We define a recursive function $\varphi : \mathbb{N}^2 \to \mathbb{N}^2$ recursively with auxiliary definitions of $\bar{k}_n, \bar{p}_n \in \mathbb{N}$ and $\theta_n : \mathbb{N} \to \mathbb{N}^2$ as follows.

$$\varphi(0, l) = \varphi'(l), \qquad \varphi(n+1, l) = \varphi''(\theta_n(l), l+1),$$
$$\bar{k}_n = \kappa(\lambda l \in \mathbb{N}.\, \varphi(n, l)), \qquad \bar{p}_n = \rho(\lambda l \in \mathbb{N}.\, \varphi(n, l)),$$
$$\theta_n(l) = \varphi(n, \max\{\psi(\bar{k}_n + 1, l + 1), \bar{p}_n).$$

It is straightforward from the definition that φ is recursive. We will show by induction that φ is total and that $(\xi_{\varphi(n,l)})$ converges to $(f^n(a))$ uniformly in n and exponentially in l as $l \to \infty$. It is trivial that $\varphi(0,l)$ is defined and $d(\xi_{\varphi(0,l)}, a) < 1/2^l$ for any $l \in \mathbb{N}$. It remains to show that if $\varphi(n,l)$ is defined and $d(\xi_{\varphi(n,l)}, f^n(a)) < 1/2^l$ for any $l \in \mathbb{N}$, then $\varphi(n+1,l)$ is defined and $d(\xi_{\varphi(n+1,l)}, f^{n+1}(a)) < 1/2^l$ for any $l \in \mathbb{N}$.

From the induction hypothesis, we obtain $\bar{B}(f^n(a), 1/2^{\bar{p}_n}) \subseteq K_{\bar{k}_n}$. Furthermore, we obtain

$$\begin{aligned} d(\xi_{\theta_n(l)}, f^n(a)) &\leq \frac{1}{2^{\max\{\psi(\bar{k}_n+1,l+1),\bar{p}_n\}}} \\ &\leq \frac{1}{2^{\bar{p}_n}}. \end{aligned}$$

Therefore, $f^n(a), \xi_{\theta_n(l)} \in K_{\bar{k}_n}$.

Meanwhile, we obtain

$$\begin{aligned} d(\xi_{\theta(l)}, f^n(a)) &\leq \frac{1}{2^{\max\{\psi(\bar{k}_n+1,l+1),\bar{p}_n\}}} \\ &\leq \frac{1}{2^{\psi(\bar{k}_n+1,l+1)}}. \end{aligned}$$

It concludes that

$$\begin{aligned} d(\xi_{\varphi(n+1,l)}, f^{n+1}(a)) &\leq d(\xi_{\varphi''(\theta_n(l),l+1)}, f(\xi_{\theta_n(l)})) + d(f(\xi_{\theta_n(l)}), f(f^n(a))) \\ &\leq \frac{1}{2^{l+1}} + \frac{1}{2^{l+1}} = \frac{1}{2^l}. \end{aligned}$$

Due to Lemma 2 (*3* $\Rightarrow$ *1*), this implies that $(f^n(a))_{n\in\mathbb{N}}$ forms a computable sequence of points. □

We are now ready to introduce the main theorem.

Theorem 1 (Effective contraction theorem). *Let $(M, d, \mathcal{S})$ be an effectively locally compact metric space. If $f : M \to M$ is a computable contraction, then* $\mathrm{Fix}\, f$ *is a computable point.*

Proof. Let L be a real such that $0 < L < 1$ and $d(f(x), f(y)) \leq Ld(x,y)$ for any $x, y \in M$. We can assume without loss of generality that L is a computable real since otherwise, we can use, instead of L, a computable real L' with $L < L' < 1$.

We start with a computable point $x_0 \in M$ and construct a sequence (x_n) by $x_{n+1} = f(x_n)$. Note that (x_n) converges to $\mathrm{Fix}\, f$. Using Lemma 3, we obtain from computability of x_0 and f that (x_n) is a computable sequence. By induction on n, we have $d(x_n, x_{n+1}) \leq L^n d(x_0, x_1)$. Hence

$$d(x_m, x_n) \leq \frac{L^m d(x_0, x_1)}{1 - L} \qquad \text{for } m < n,$$

which implies that (x_n) is an effective Cauchy sequence. Using Lemma 1, we obtain that $\operatorname{Fix} f$ is a computable point. □

The following form of the effective contraction theorem is often more convenient to check the computability of the fixed point.

Corollary 1. *Let $(M, d, \mathcal{S})$ be an effectively locally compact metric space with (K_k) and $(\xi_{k,i})$. If $f : M \to M$ is a contraction that maps $(\xi_{k,i})$ to a computable sequence, then* $\operatorname{Fix} f$ *is a computable point.*

Proof. It suffices to show that f is a computable function. From f being a contraction, it follows immediately that f is effectively uniformly continuous. What remains to show is that f is sequentially computable.

Let (x_n) be a computable sequence. Using Lemma 2, we obtain there exists a recursive function $\varphi : \mathbb{N}^2 \to \mathbb{N}^2$ such that $(\xi_{\varphi(n,l)})$ converges to (x_n) effectively in n and l as $l \to \infty$. Since $(f(\xi_{\varphi(n,l)}))$ is an effectively selected subsequence of $(f(\xi_{k,i}))$, it holds that $(f(\xi_{\varphi(n,l)}))$ is a computable sequence. Since f is a contraction, $(f(\xi_{\varphi(n,l)}))$ converges to $(f(x_n))$ effectively in n and l as $l \to \infty$. Due to Lemma 2, it follows that $(f(x_n))$ is a computable sequence. □

3.2 Application

3.2.1 Computability of Self-similar Sets

We denote by $\|\text{-}\|$ the Euclidean norm. For a nonempty subset S of $\mathbb{R}^q$, the function $\underline{d}_S : \mathbb{R}^q \to [0, +\infty)$ is defined by $\underline{d}_S(x) = \inf_{a \in S} \|x - a\|$.

Let $\varphi_1, \ldots, \varphi_m : \mathbb{R}^q \to \mathbb{R}^q$ be contractions. Then there exists a unique nonempty compact subset X of $\mathbb{R}^q$ such that

$$X = \varphi_1(X) \cup \cdots \cup \varphi_m(X). \tag{3.1}$$

Hutchinson [8] first proved this theorem by using the contraction theorem on the space of nonempty compact subsets of a Euclidean space with the Hausdorff metric. Hata [6] generalized the theorem. Kamo and Kawamura [10] added the viewpoint of computability; they have shown that if all of $\varphi_1, \ldots, \varphi_m$ are computable, then the unique solution X of the equation (3.1) satisfies that $\underline{d}_X$ is a computable function. We will rewrite Kamo and Kawamura's proof by using the effective contraction theorem.

Let $\mathcal{K}(\mathbb{R}^q)$ denote the set of all nonempty compact subsets of $\mathbb{R}^q$. A complete metric on $\mathcal{K}(\mathbb{R}^q)$ known as the *Hausdorff metric* d_{H} is defined by:

$$d_{\mathrm{H}}(K, L) = \max\{\sup_{a \in K} \inf_{b \in L} \|a - b\|, \sup_{b \in L} \inf_{a \in K} \|b - a\|\}.$$

Lemma 4. *For any $K, L \in \mathcal{K}(\mathbb{R}^q)$ and $x \in \mathbb{R}^q$, it holds $|\underline{d}_K(x) - \underline{d}_L(x)| \leq d_{\mathrm{H}}(K, L)$.*

Proof. Some manipulation of sup's and inf's yields that

$$\begin{aligned}\underline{d}_K(x) - \underline{d}_L(x) &= \sup_{b\in L} \inf_{a\in K} (\|x-a\| - \|x-b\|) \\ &\leq \sup_{b\in L} \inf_{a\in K} \|b-a\|\end{aligned}$$

By exchanging K and L, we obtain also that

$$\underline{d}_L(x) - \underline{d}_K(x) \leq \sup_{a\in K} \inf_{b\in L} \|a-b\|.$$

Therefore, $|\underline{d}_K(x) - \underline{d}_L(x)| \leq d_{\mathrm{H}}(K, L)$. □

We denote by $\mathcal{S}_{\mathrm{H}}$ a set of sequences over $\mathcal{K}(\mathbb{R}^q)$ such that $(K_n) \in \mathcal{S}_{\mathrm{H}}$ iff $(\underline{d}_{K_n})$ is a computable sequence of functions. We define $\mathfrak{K}_k = \{K \in \mathcal{K}(\mathbb{R}^q) \mid K \subseteq \bar{B}(0, k+1)\}$. We call $K \in \mathcal{K}(\mathbb{R}^q)$ a rational finite set if K is a finite set of rational points. Let $(\Xi_{k,i})_{k,i}$ be an effective enumeration of all rational finite sets in $\mathcal{K}(\mathbb{R}^q)$ such that $(\Xi_{k,i})_i$ is an effective enumeration of all rational finite sets in $\mathfrak{K}_k$ for each k.

Lemma 5. *If (K_m) and (L_n) are effectively selected subsequences of $(\Xi_{k,i})$, then $(\sup_{a\in K_m} \underline{d}_{L_n}(a))_{m,n}$ forms a computable double sequence of reals.*

The proof is straightforward from the effectiveness of enumeration of $(\Xi_{k,i})$.

Proposition 2. *$(\mathcal{K}(\mathbb{R}^q), d_{\mathrm{H}}, \mathcal{S}_{\mathrm{H}})$ is an effectively locally compact metric space with $(\mathfrak{K}_k)$ and $(\Xi_{k,i})$.*

Proof. It is a well-known property of the Hausdorff metric that $(\mathcal{K}(\mathbb{R}^q), d_{\mathrm{H}})$ is a complete metric space.

Next, we verify that $\mathcal{S}_{\mathrm{H}}$ is a computability structure on $(\mathcal{K}(\mathbb{R}^q), d_{\mathrm{H}})$. Lemma 5 implies that if $(K_m), (L_n) \in \mathcal{S}$, then $(d_{\mathrm{H}}(K_m, L_n))_{m,n}$ forms a computable sequence of reals, i.e., the metric axiom holds. Checking the subsequence axiom is straightforward from the properties of computable real-valued functions on $\mathbb{R}^q$. Lemma 4 implies that if $(K_{n,k})$ converges to (K_n) effectively in n and k as $k \to \infty$, then $(\underline{d}_{K_{n,k}})$ uniformly converges to $(\underline{d}_{K_n})$ effectively in n and k as $k \to \infty$. Hence the limit axiom holds.

Finally, we verify that $(\mathcal{K}(\mathbb{R}^q), d_{\mathrm{H}}, \mathcal{S}_{\mathrm{H}})$ is an effectively locally compact metric space with $(\mathfrak{K}_k)$ and $(\Xi_{k,i})$. It follows immediately from the definition that $\overline{\{\Xi_{k,i} \mid i \in \mathbb{N}\}} = \mathfrak{K}_k$. It follows from the effectiveness of enumeration of $(\Xi_{k,i})$ and denseness of $(\Xi_{k,i})$ in $\mathfrak{K}_k$ that there exists a recursive function $\psi : \mathbb{N}^2 \to \mathbb{N}$ such that for any k, it holds $\bigcup_{i<\psi(k,p)} B(\Xi_{k,i}, 1/2^p) \supseteq \mathfrak{K}_k$. Since $K \subseteq \bar{B}(0, k+1)$ and $d_{\mathrm{H}}(K, K') \leq 1$ imply $K' \subseteq \bar{B}(0, k+2)$, we have for any $\boldsymbol{i} \in \mathbb{N}^{\mathbb{N}}$ and $k \in \mathbb{N}$, if $(\Xi_{k,\boldsymbol{i}[j]})$ converges exponentially in j as $j \to \infty$ and $\lim_{j\to\infty} \xi_{k,\boldsymbol{i}[j]} \in \mathfrak{K}_k$, then $B(\lim_{j\to\infty} \Xi_{k,\boldsymbol{i}[j]}, 1) \subseteq \mathfrak{K}_{k+1}$. $\square$

The following lemma is used in the proof of Hutchinson and Hata's Theorem. Refer to [8] or [6] for detail.

Lemma 6. *For $\varphi_1, \ldots, \varphi_m : \mathbb{R}^q \to \mathbb{R}^q$, define $\Phi : \mathcal{K}(\mathbb{R}^q) \to \mathcal{K}(\mathbb{R}^q)$ by:*

$$\Phi(X) = \varphi_1(X) \cup \cdots \cup \varphi_m(X). \tag{3.2}$$

If all of $\varphi_1, \ldots, \varphi_m$ are contractions, then Φ is a contraction on $\mathcal{K}(\mathbb{R}^q)$.

In addition, we will use the following lemma.

Lemma 7. *For $\varphi_1, \ldots, \varphi_m : \mathbb{R}^q \to \mathbb{R}^q$, define $\Phi : \mathcal{K}(\mathbb{R}^q) \to \mathcal{K}(\mathbb{R}^q)$ by (3.2) in Lemma 6. If all of $\varphi_1, \ldots, \varphi_m$ are computable functions, then Φ maps $(\Xi_{k,i})$ to a computable sequence.*

Proof. Since $\Xi_{k,i}$ is a finite set, so is $\Phi(\Xi_{k,i})$. Thus

$$\underline{d}_{\Phi(\Xi_{k,i})}(x) = \min \bigcup_{a \in \Xi_{k,i}} \{\|\varphi_1(a) - x\|, \ldots, \|\varphi_m(a) - x\|\}. \tag{3.3}$$

The right-hand side is the minimum of the values of finitely many computable functions and the minimization is uniform on k and i. Therefore $(\underline{d}_{\Phi(\Xi_{k,i})})$ is a computable sequence of functions, i.e., $(\Phi(\Xi_{k,i}))$ is computable. $\square$

Now we prove the result on computability of self-similar sets by using the effective contraction theorem. Let $\varphi_1, \ldots, \varphi_m : \mathbb{R}^q \to \mathbb{R}^q$ be computable contractions. Define $\Phi : \mathcal{K}(\mathbb{R}^q) \to \mathcal{K}(\mathbb{R}^q)$ by (3.2) in Lemma 6. By using Lemmata 6 and 7, we obtain that $\operatorname{Fix} \Phi$ is a computable point on $\mathcal{K}(\mathbb{R}^q)$, i.e., the unique compact nonempty solution X of the equation (3.1) satisfies that $\underline{d}_X$ is a computable function.

Theorem 2 (Kamo and Kawamura). *Let $\varphi_1, \ldots, \varphi_m : \mathbb{R}^q \to \mathbb{R}^q$ be computable contractions. Then there exists a unique nonempty compact subset X of $\mathbb{R}^q$ such that*

$$X = \varphi_1(X) \cup \cdots \cup \varphi_m(X).$$

For such an X, it holds that $\underline{d}_X$ is a computable function.

Consider an extension of Hutchinson and Hata's Theorem to systems of fixed-point equations [1]. During this consideration, we implicitly declare the index j ranging over $1, \ldots, p$ and we omit the phrase "for any $j = 1, \ldots, p$".

Let $\varphi_{j11}, \ldots, \varphi_{j1m_{j1}}, \ldots, \varphi_{jp1}, \ldots, \varphi_{jpm_{jp}} : \mathbb{R}^q \to \mathbb{R}^q$ be contractions. Let each $K_j \subset \mathbb{R}^q$ be a compact subset. Then there exists a unique p-tuple $(X_1, \ldots, X_p)$ of nonempty compact subsets of $\mathbb{R}^q$ such that

$$X_j = K_j \cup (\varphi_{j11}(X_1) \cup \cdots \cup \varphi_{j1m_{j1}}(X_1)) \cup \cdots \cup (\varphi_{jp1}(X_p) \cup \cdots \cup \varphi_{jpm_{jp}}(X_p)).$$

We effectivize also this extended theorem.

Theorem 3. *Let $\varphi_{j11}, \ldots, \varphi_{j1m_{j1}}, \ldots, \varphi_{jp1}, \ldots, \varphi_{jpm_{jp}} : \mathbb{R}^q \to \mathbb{R}^q$ be computable contractions. Let $K_j \subset \mathbb{R}^q$ be a compact subset such that K_j is empty or $\underline{d}_{K_j}$ is a computable function. Then there exists a unique p-tuple $(X_1, \ldots, X_p)$ of nonempty compact subsets of $\mathbb{R}^q$ such that*

$$X_j = K_j \cup (\varphi_{j11}(X_1) \cup \cdots \cup \varphi_{j1m_{j1}}(X_1)) \cup \cdots \cup (\varphi_{jp1}(X_p) \cup \cdots \cup \varphi_{jpm_{jp}}(X_p)).$$

For such a p-tuple $(X_1, \ldots, X_p)$, each $\underline{d}_{X_j}$ is a computable function.

The proof is analogous to that for Theorem 2. Use $\mathcal{K}(\mathbb{R}^q)^p$ instead of $\mathcal{K}(\mathbb{R}^q)$ as an effectively locally compact space. Use the defining formula

$$\begin{aligned}
&\Phi(X_1, \ldots, X_p) = (\Phi_1(X_1, \ldots, X_p), \ldots, \Phi_p(X_1, \ldots, X_p)),\\
&\Phi_j(X_1, \ldots, X_p)\\
&= K_j \cup (\varphi_{j11}(X_1) \cup \cdots \cup \varphi_{j1m_{j1}}(X_1)) \cup \cdots \cup (\varphi_{jp1}(X_p) \cup \cdots \cup \varphi_{jpm_{jp}}(X_p))
\end{aligned}$$

instead of (3.2) in Lemmata 6 and 7 to construct a computable contraction Φ on $\mathbb{R}^p$. The important intermediate result corresponding to (3.3) in the proof of Lemma 7 is that

$$\begin{aligned}
&\underline{d}_{\Phi_j(\Xi_{k_1,i_1}, \ldots, \Xi_{k_p,i_p})}(x)\\
&\quad = \min\Bigg(S_j \cup \bigcup_{a \in \Xi_{k_1,i_1}} \{\|\varphi_{j11}(a) - x\|, \ldots, \|\varphi_{j1m_{j1}}(a) - x\|\}\\
&\qquad\qquad \cup \cdots \cup \bigcup_{a \in \Xi_{k_p,i_p}} \{\|\varphi_{jp1}(a) - x\|, \ldots, \|\varphi_{jpm_{jp}}(a) - x\|\}\Bigg)\\
&\qquad\qquad\qquad \text{where } S_j = \begin{cases} \emptyset & \text{if } K_j = \emptyset, \\ \{\underline{d}_{K_j}(x)\} & \text{otherwise.} \end{cases}
\end{aligned}$$

Note that Theorem 2 is a special case of Theorem 3 with $p = 1$ and $K_1 = \emptyset$.

Chapter 4

Dini's Theorem

If a sequence of real-valued continuous functions on a compact space converges pointwise monotonically to a continuous function, then the sequence converges uniformly to the function. It is called *Dini's theorem* and one of the fundamental theorems in functional analysis and general topology.

From the viewpoint of computability, the question arises: whether we can effectivize Dini's theorem, in other words, whether there is a theorem which is a Dini's theorem with some topological concepts replaced by their computational counterparts. In this chapter, we show a positive answer to this question in the case of metric spaces, more precisely, the theorem that if a computable sequence of real-valued functions on an effectively compact metric space converges pointwise monotonically to a computable function, then the sequence converges effectively uniformly to the function.

Meanwhile, if a computable sequence of real numbers converges monotonically to a computable real number, then the sequence converges effectively to the real number. It is called the *monotonic convergence theorem* [14]. The main theorem in this chapter is not only an effectivization of Dini's theorem but also an extension of the monotonic convergence theorem to real-valued continuous functions on effectively compact metric spaces. The monotonic convergence theorem is considered a special case of our theorem on $C(\{0\})$, the space of all real-valued continuous functions on a singleton.

4.1 Covers

First, we show two propositions with no assumptions on computability.

Proposition 3. *Let (M, d) be a metric space. Let $K \subset M$ be a nonempty compact subset with $(e_l) \in M^\omega$ and $\gamma : \mathbb{N} \to \mathbb{N}$ such that $K = \overline{\{e_l \mid l \in \mathbb{N}\}}$*

and $(\forall i)\ K \subset \bigcup_{l=0}^{\gamma(i)} B(e_l, 1/2^i)$*. Then, for any finite sequence of open balls* $(B(x_k, r_k))_{k=0,\dots,m}$*, the following holds:*

$$K \subset \bigcup_{k=0}^{m} B(x_k, r_k) \iff (\exists i)(\forall l \leq \gamma(i))(\exists k \leq m)\ d(x_k, e_l) + \frac{1}{2^i} < r_k.$$

Proof. ($\Longleftarrow$) Suppose $y \in K$. Then, $(\exists l \leq \gamma(i))\ d(y, e_l) < 1/2^i$. It follows that

$$(\exists i)(\exists l \leq \gamma(i))(\exists k \leq m)\ \left[d(y, e_l) < \frac{1}{2^i}\ \wedge\ d(x_k, e_l) + \frac{1}{2^i} < r_k \right].$$

This implies $(\exists k \leq m)\ d(y, x_k) < r_k$, i.e., $y \in \bigcup_{k=0}^{m} B(x_k, r_k)$. This shows $K \subset \bigcup_{k=0}^{m} B(x_k, r_k)$.

($\Longrightarrow$) We will prove the contraposition. The negation of the conclusion is:

$$(\forall i)(\exists l \leq \gamma(i))(\forall k \leq m) \quad d(x_k, e_l) + \frac{1}{2^i} \geq r_k.$$

By choosing an l for each i, we obtain a function $\gamma' : \mathbb{N} \to \mathbb{N}$ such that

$$(\forall i)(\forall k \leq m) \quad d(x_k, e_{\gamma'(i)}) + \frac{1}{2^i} \geq r_k.$$

Since K is sequentially compact, there exists a subsequence $(e_{\gamma'(\theta(i))})$ that converges to a point in K. Let y be the limit. A simple manipulation of limits yields $(\forall k \leq m)\ d(x_k, y) \geq r_k$, i.e., $y \notin \bigcup_{k=0}^{m} B(x_k, r_k)$. This shows $K \not\subset \bigcup_{k=0}^{m} B(x_k, r_k)$. □

Proposition 4. *Let* (M, d) *be a separable metric space with a dense, at most countable subset* $\{e_l \mid l \in \mathbb{N}\}$*. Let* a *be a real number such that* $a \geq 1$*. Let* (ε_i) *be a sequence of positive real numbers converging to* 0*. Then,*

$$B(x, r) = \bigcup_{\substack{l, i \in \mathbb{N}, \\ d(x, e_l) + a\varepsilon_i < r}} B(e_l, \varepsilon_i).$$

Proof. Comparison of the distance between the centers with the difference between the radii yields $B(x, r) \supset B(e_l, \varepsilon_i)$ if $d(x, e_l) + a\varepsilon_i < r$. This implies $B(x, r) \supset \bigcup_{l, i \in \mathbb{N},\ d(x, e_l) + a\varepsilon_i < r} B(e_l, \varepsilon_i)$.

Suppose $y \in B(x, r)$. There exists an ε_i such that $d(x, y) < r - (a + 1)\varepsilon_i$. Furthermore, there exists an e_l such that $d(y, e_l) < \varepsilon_i$. From these two inequalities as well as $d(x, e_l) \leq d(x, y) + d(y, e_l)$, we obtain $d(x, e_l) + a\varepsilon_i < r$. Therefore $y \in \bigcup_{l,i \in \mathbb{N},\, d(x,e_l)+a\varepsilon_i<r} B(e_l, \varepsilon_i)$. This shows

$$B(x, r) \subset \bigcup_{\substack{l,i \in \mathbb{N}, \\ d(x,e_l)+a\varepsilon_i<r}} B(e_l, \varepsilon_i).$$

□

Next, we show two lemmata. Lemma 8 is on a consequence of effective compactness. Lemma 9 is on another characterization of computable real-valued functions.

Lemma 8. *Let $(M, d, \mathcal{S})$ be a metric space with a computability structure. For any effectively compact subset K of M and any computable double sequence $(B(x_{n,k}, r_{n,k}))$ of open balls in M, there exists a recursive partial function $\alpha : \mathbb{N} \rightharpoonup \mathbb{N}$ such that $\alpha(n)$ is defined and $K \subset \bigcup_{k=0}^{\alpha(n)} B(x_{n,k}, r_{n,k})$ holds if $K \subset \bigcup_{k=0}^{\infty} B(x_{n,k}, r_{n,k})$, and $\alpha(n)$ is undefined otherwise.*

Proof. We choose a computable sequence (e_l) and a recursive function $\gamma : \mathbb{N} \to \mathbb{N}$ such that $K = \overline{\{e_l \mid l \in \mathbb{N}\}}$ and $(\forall i)\ K \subset \bigcup_{l=0}^{\gamma(i)} B(e_l, 1/2^i)$. By applying Proposition 3 to each finite sequence $(B(x_{n,k}, r_{n,k}))_{k=0,\dots,m}$ for $n = 0, 1, 2, \dots$, we obtain

$$K \subset \bigcup_{k=0}^{m} B(x_{n,k}, r_{n,k}) \iff (\exists i)(\forall l \leq \gamma(i))(\exists k \leq m)\ d(x_{n,k}, e_l) + \frac{1}{2^i} < r_{n,k}.$$

It is clear that the right-hand side is recursively enumerable in n and m. Therefore, there exists a primitive recursive predicate φ on $\mathbb{N}^3$ such that $(\exists i')\varphi(n, m, i')$ iff $K \subset \bigcup_{k=0}^{m} B(x_{n,k}, r_{n,k})$. Hence, we can construct α by $\alpha(n) \simeq (\min\{m' \mid \varphi(n, (m')^2_1, (m')^2_2)\})^2_1$. □

Lemma 8 corresponds to "$\delta'_{\mathrm{range}} \leq \delta_{\mathrm{Haine-Borel}}|^{\mathcal{K}^*}$" shown by Brattka and Presser in [2]. The proof here is essentially the same as that in [2] with some correction for a minor error.

Lemma 9. *Let $(M, d, \mathcal{S})$ be an effectively compact metric space. Let $(e_l) \in \mathcal{S}$ be dense in M. For any sequence (f_n) of real-valued functions on M, the following two conditions are equivalent.*

1. *(f_n) is a computable sequence of functions.*

2. *There exists a recursively enumerable set* $S \subset \mathbb{N} \times \mathbb{Q} \times \mathbb{Q}^+ \times \mathbb{N} \times \mathbb{N}$ *such that*

$$(\forall n \in \mathbb{N})(\forall c \in \mathbb{Q})(\forall r \in \mathbb{Q}^+)$$
$$f_n^{-1}((c-r,\, c+r)) = \bigcup_{(l,i)\in S(n,c,r)} B(e_l, 1/2^i).$$

Proof. [(1)⇒(2)]. Let $\alpha : \mathbb{N}^2 \to \mathbb{N}$ be an effective modulus of continuity of (f_n). Let $S \subset \mathbb{N} \times \mathbb{Q} \times \mathbb{Q}^+ \times \mathbb{N} \times \mathbb{N}$ be the set defined by

$$(n, c, r, l, i) \in S$$
$$\iff (\exists j \in \mathbb{N}) \left[i = \alpha(n, j) \;\wedge\; c - r < f_n(e_l) - \frac{1}{2^j} \;\wedge\; f_n(e_l) + \frac{1}{2^j} < c + r \right].$$

Since α is a recursive function and $(f_n(e_l))$ is a computable double sequence of real numbers, it follows immediately from the definition of S that S is a recursively enumerable set.

Let $(l, i) \in S(n, c, r)$. Then we have, for some $j \in \mathbb{N}$,

$$\begin{aligned} f_n(B(e_l, 1/2^i)) &= f_n(B(e_l, 1/2^{\alpha(n,j)})) \\ &\subset (f_n(e_l) - 1/2^j,\, f_n(e_l) + 1/2^j) \\ &\subset (c - r,\, c + r). \end{aligned}$$

Therefore, $B(e_l, 1/2^i) \subset f_n^{-1}((c-r,\, c+r))$. This shows $f_n^{-1}((c-r,\, c+r)) \supseteq \bigcup_{(l,i)\in S(n,c,r)} B(e_l, 1/2^i)$.

Suppose $x \in f_n^{-1}((c-r,\, c+r))$, i.e., $|f_n(x) - c| < r$. Then, for some $j \in \mathbb{N}$, it holds that $2/2^j \leq r - |f_n(x) - c|$. For such a j, there exists an e_l such that $d(x, e_l) < 1/2^{\alpha(n,j)}$. Hence $|f_n(e_l) - f_n(x)| < 1/2^j$. By using $f_n(e_l) - 1/2^j < f_n(x)$, we obtain

$$f_n(e_l) + \frac{1}{2^j} < f_n(x) + \frac{2}{2^j} \leq f_n(x) + r - |f_n(x) - c| \leq c + r.$$

Analogously, by using $f_n(e_l) + 1/2^j > f_n(x)$, we obtain $f_n(e_l) - 1/2^j > c - r$. The conjunction of the obtained two inequalities implies $(n, c, r, l, \alpha(n, j)) \in S$. Therefore, $x \in B(e_l, 1/2^i)$ for some $(l, i) \in S(n, c, r)$. This shows $f_n^{-1}((c-r,\, c+r)) \subset \bigcup_{(l,i)\in S(n,c,r)} B(e_l, 1/2^i)$.

[(2)⇒(1)] (*Sequential computability*). Suppose $(x_k) \in \mathcal{S}$. From (2), we obtain

$$|f_n(x_k) - c| < \frac{1}{2^j} \iff (\exists l)(\exists i) \left[(n, c, 1/2^j, l, i) \in S \;\wedge\; d(x_k, e_l) < \frac{1}{2^i} \right].$$

Hence $\{(n,k,j,c) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N}\times\mathbb{Q} \mid |f_n(x_k)-c| < 1/2^j\}$ is a recursively enumerable set. Meanwhile, $(\forall n)(\forall k)(\forall j)(\exists c \in \mathbb{Q})\ |f_n(x_k)-c| < 1/2^j$ holds since $\mathbb{Q}$ is dense in $\mathbb{R}$. Therefore, there exists a computable triple sequence of rational numbers $(c_{n,k,j})$ such that $(\forall n)(\forall k)(\forall j)\ |f_n(x_k)-c_{n,k,j}| < 1/2^j$, i.e., $(f_n(x_k))_{n,k}$ is a computable double sequence of real numbers.

(*Effective uniform continuity*). Using Proposition 4, we obtain that

$$B(e_l, 1/2^i) = \bigcup_{\substack{l',i'\in\mathbb{N},\\ d(e_l,e_{l'})+2/2^{i'}<1/2^i}} B(e_{l'}, 1/2^{i'}).$$

It is clear that $d(e_l,e_{l'})+2/2^{i'} < 1/2^i$ is a recursively enumerable predicate of l,i,l',i'. It is also clear that $(\forall l)(\forall i)(\exists l')(\exists i')\ d(e_l,e_{l'})+2/2^{i'} < 1/2^i$ holds. Hence there exist recursive functions $\sigma,\rho : \mathbb{N}^3 \to \mathbb{N}$ such that for any $l,i \in \mathbb{N}$,

$$\begin{aligned}\{(l,i,l',i') \in \mathbb{N}\times\mathbb{N}\times\mathbb{N}\times\mathbb{N} \mid d(e_l,e_{l'})+2/2^{i'} < 1/2^i\}&\\ = \{(l,i,\sigma(l,i,k),\rho(l,i,k)) \mid k \in \mathbb{N}\}&.\end{aligned}$$

Therefore, for any $l,i \in \mathbb{N}$,

$$B(e_l, 1/2^i) = \bigcup_{k=0}^{\infty} B(e_{\sigma(l,i,k)}, 1/2^{\rho(l,i,k)}) \tag{4.1}$$

and

$$(\forall k)\quad d(e_l, e_{\sigma(l,i,k)}) + \frac{2}{2^{\rho(l,i,k)}} < \frac{1}{2^i}. \tag{4.2}$$

Since S is a recursively enumerable set, so is $\{(n,j,l,i,c) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N}\times\mathbb{N}\times\mathbb{Q} \mid (n,c,1/2^{j+1},l,i) \in S\}$. Meanwhile, since

$$M = \bigcup_{c\in\mathbb{Q}} f_n^{-1}((c-1/2^{j+1},\ c+1/2^{j+1})),$$

it holds

$$M = \bigcup_{c\in\mathbb{Q}} \bigcup_{(l,i)\in S(n,c,1/2^{j+1})} B(e_l, 1/2^i).$$

Hence $(\forall n)(\forall j)(\exists l)(\exists i)(\exists c \in \mathbb{Q})\ (n,c,1/2^{j+1},l,i) \in S$. Therefore, there exist recursive functions $\sigma',\rho' : \mathbb{N}^3 \to \mathbb{N}$ and a computable triple sequence of rational numbers $(c_{n,j,k})$ such that for any $n,j \in \mathbb{N}$,

$$\begin{aligned}\{(n,j,l,i,c) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N}\times\mathbb{N}\times\mathbb{Q} \mid (n,c,1/2^{j+1},l,i) \in S\}&\\ = \{(n,j,\sigma'(n,j,k),\rho'(n,j,k),c_{n,j,k}) \mid k \in \mathbb{N}\}&.\end{aligned}$$

Thus, for any $n, j \in \mathbb{N}$,

$$M = \bigcup_{k=0}^{\infty} B(e_{\sigma'(n,j,k)}, 1/2^{\rho'(n,j,k)}) \tag{4.3}$$

and

$$(\forall k) \quad f_n(B(e_{\sigma'(n,j,k)}, 1/2^{\rho'(n,j,k)})) \subset (c_{n,j,k} - 1/2^{j+1},\, c_{n,j,k} + 1/2^{j+1}). \tag{4.4}$$

From (4.1) and (4.3), we obtain

$$M = \bigcup_{k=0}^{\infty} B(e_{\sigma(\sigma'(n,j,(k)^2_1),\rho'(n,j,(k)^2_1),(k)^2_2)}, 1/2^{\rho(\sigma'(n,j,(k)^2_1),\rho'(n,j,(k)^2_1),(k)^2_2)}).$$

Application of Lemma 8 yields that there exists a recursive function $\gamma : \mathbb{N}^2 \to \mathbb{N}$ such that

$$M = \bigcup_{k=0}^{\gamma(n,j)} B(e_{\sigma(\sigma'(n,j,(k)^2_1),\rho'(n,j,(k)^2_1),(k)^2_2)}, 1/2^{\rho(\sigma'(n,j,(k)^2_1),\rho'(n,j,(k)^2_1),(k)^2_2)}).$$

Let $\alpha : \mathbb{N}^2 \to \mathbb{N}$ be a recursive function defined by

$$\alpha(n,j) = \max_{k \le \gamma(n,j)} \rho(\sigma'(n,j,(k)^2_1), \rho'(n,j,(k)^2_1), (k)^2_2).$$

Suppose points $x, y \in M$ satisfy $d(x,y) < 1/2^{\alpha(n,j)}$. Then there exists some $k \le \gamma(n,j)$ such that

$$d(x, e_{\sigma(\sigma'(n,j,(k)^2_1),\rho'(n,j,(k)^2_1),(k)^2_2)}) < \frac{1}{2^{\rho(\sigma'(n,j,(k)^2_1),\rho'(n,j,(k)^2_1),(k)^2_2)}}.$$

With such an index k,

$$\begin{aligned}
&d(x, e_{\sigma'(n,j,(k)^2_1)}) \\
&\le d(e_{\sigma'(n,j,(k)^2_1)}, e_{\sigma(\sigma'(n,j,(k)^2_1),\rho'(n,j,(k)^2_1),(k)^2_2)}) \\
&\qquad + d(x, e_{\sigma(\sigma'(n,j,(k)^2_1),\rho'(n,j,(k)^2_1),(k)^2_2)}) \\
&< d(e_{\sigma'(n,j,(k)^2_1)}, e_{\sigma(\sigma'(n,j,(k)^2_1),\rho'(n,j,(k)^2_1),(k)^2_2)}) + \frac{1}{2^{\rho(\sigma'(n,j,(k)^2_1),\rho'(n,j,(k)^2_1),(k)^2_2)}} \\
&< \frac{1}{2^{\rho'(n,j,(k)^2_1)}} - \frac{1}{2^{\rho(\sigma'(n,j,(k)^2_1),\rho'(n,j,(k)^2_1),(k)^2_2)}}.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
d(y, e_{\sigma'(n,j,(k)^2_1)}) &\le d(x, e_{\sigma'(n,j,(k)^2_1)}) + d(x,y) \\
&< \frac{1}{2^{\rho'(n,j,(k)^2_1)}} - \frac{1}{2^{\rho(\sigma'(n,j,(k)^2_1),\rho'(n,j,(k)^2_1),(k)^2_2)}} + \frac{1}{2^{\alpha(n,j)}} \\
&\le \frac{1}{2^{\rho'(n,j,(k)^2_1)}}.
\end{aligned}$$

Therefore, $x, y \in B(e_{\sigma'(n,j,(k)^2_1)}, 1/2^{\rho'(n,j,(k)^2_1)})$. With (4.4), this implies

$$f_n(x), f_n(y) \in (c_{n,j,(k)^2_1} - 1/2^{j+1},\ c_{n,j,(k)^2_1} + 1/2^{j+1}).$$

Hence $|f(x) - f(y)| < 1/2^j$. This shows that α is an effective modulus of continuity of (f_n). $\square$

The condition (2) in Lemma 9 is equivalent to δ_6-computability defined by Weihrauch in [18]. Lemma 9 shows that for a real-valued function on an effectively compact metric space, computability defined by Mori, Tsujii, and Yasugi in [13] and used in this paper is equivalent to δ_6-computability.

4.2 Effective Dini's Theorem

Now we are ready to show the effective Dini's theorem.

Theorem 4. *Let $(M, d, \mathcal{S})$ be an effectively compact metric space. Let (f_n) be a computable sequence of real-valued functions on M and f a computable real-valued function on M. If f_n converges pointwise monotonically to f as $n \to \infty$, then f_n converges effectively uniformly to f.*

Proof. Let (e_l) be a computable sequence dense in M.

Let $U_{j,n} = (f_n - f)^{-1}((-1/2^j, 1/2^j))$. Since f_n converges pointwise to f, it holds that $(\forall j)\ M = \bigcup_{n=0}^{\infty} U_{j,n}$. Due to Lemma 9, there exists a recursively enumerable set $S \subset \mathbb{N}^4$ such that $(\forall j)(\forall n)\ U_{j,n} = \bigcup_{(i,l)\in S(j,n)} B(e_l, 1/2^i)$. From these two equalities, we obtain

$$(\forall j) \quad M = \bigcup_{(n,i,l)\in S(j)} B(e_l, 1/2^i).$$

This implies $(\forall j)\ S(j) \neq \emptyset$. Therefore, there exist recursive functions $\theta, \rho, \sigma : \mathbb{N}^2 \to \mathbb{N}$ such that $S = \{(j, \theta(j,k), \rho(j,k), \sigma(j,k)) \mid j, k \in \mathbb{N}\}$. Using these functions, we can rewrite the formula above as follows:

$$(\forall j) \quad M = \bigcup_{k=0}^{\infty} B(e_{\sigma(j,k)}, 1/2^{\rho(j,k)}).$$

Since M itself is a compact subset of M, application of Lemma 8 yields that there exists a recursive total function $\beta : \mathbb{N} \to \mathbb{N}$ such that

$$(\forall j) \quad M = \bigcup_{k=0}^{\beta(j)} B(e_{\sigma(j,k)}, 1/2^{\rho(j,k)}).$$

Since $B(e_{\sigma(j,k)}, 1/2^{\rho(j,k)}) \subset U_{j,\theta(j,k)}$, this implies $(\forall j)\ M = \bigcup_{k=0}^{\beta(j)} U_{j,\theta(j,k)}$. Since f_n converges monotonically to f, it holds that $U_{j,n} \subset U_{j,n'}$ if $n \leq n'$. Let $\alpha : \mathbb{N} \to \mathbb{N}$ be a recursive function defined by $\alpha(j) = \max_{k \leq \beta(j)} \theta(j,k)$. We have $(\forall j)(\forall n \geq \alpha(j))\ M = U_{j,n}$, which is equivalent to:

$$(\forall j)(\forall n \geq \alpha(j))(\forall x \in M) \quad |f_n(x) - f(x)| < \frac{1}{2^j}.$$

Thus f_n converges effectively uniformly to f as $n \to \infty$. $\square$

Chapter 5

Representation-based approach to computable analysis

In this chapter, we summarize definitions used in the next chapter. In §5.1, we define computability on metric spaces by the representation-based approach. In §5.2, we show some examples.

5.1 Computability in computable metric space

We denote a standard effective enumeration of rational numbers by $\nu_{\mathbb{Q}}$.

Definition 11. (X, d_X, A, ν_A) is a *computable metric space* if:

1. (X, d_X) is a metric space,
2. A is a dense subset of (X, d_X),
3. ν_A is an enumeration of A, i.e. a surjection from $\mathbb{N}$ to A, and,
4. the set $\{\langle i, j, k, l\rangle : \nu_{\mathbb{Q}}(k) < d_X(\nu_A(i), \nu_A(j)) < \nu_{\mathbb{Q}}(l)\}$ is recursively enumerable.

A computable metric space defined above is a computable metric space such that $D_>$ is recursively enumerable in [18].

Definition 12. Let (X, d_X, A, ν_A) be a computable metric space. The *Cauchy representation* $\delta_X :\subseteq \mathbb{N}^\omega \to X$ is defined by

$$\delta_X(p) = x \iff (\forall i)(\forall j > i)\, d(\nu(p[i]), \nu(p[j])) \leq \frac{1}{2^i} \text{ and } \lim_{i\to\infty} \nu(p[i]) = x.$$

A point $x \in X$ is *computable* if there exists a recursive sequence $p \in \mathbb{N}^\omega$ such that $\delta_X(p) = x$.

The Cauchy representation is the representation δ_3 in Definition 2.4 in [18]. It is also the normed Cauchy representation in [11][17].

For a computable metric space (X, d_X, A, ν_A), we can construct a computable function $\rho_X :\subseteq \mathbb{N}^\omega \times \mathbb{N}^\omega \to \mathbb{N}^\omega$ such that for any $p, q \in \mathrm{dom}(\delta_X)$, it holds that $\delta_\mathbb{R}(\rho_X(p,q)) = d_X(\delta_X(p), \delta_X(q))$. In other words, we can translate the metric d_X to a computable function ρ_X by using δ_X. Throughout this chapter and the following chapter, ρ_X denotes such a function.

Definition 13. Let (X, d_X, A, ν_A) and (Y, d_Y, B, ν_B) be computable metric spaces. A function $f : X \to Y$ is *computable* if there exists a computable function $F : \mathbb{N}^\omega \to \mathbb{N}^\omega$ such that for any $p \in \mathrm{dom}(\delta_X)$, it holds that $\delta_Y(F(p)) = f(\delta_X(p))$.

Computability defined above is equivalent to δ_5-computability in Definition 3.1 in [18].

We often abbreviate $\nu_A(n)$ as n^A, $\nu_A \circ f$ as f^A, $\delta_X(p)$ as p^X, and $\delta_X \circ F$ as F^X. We use abbreviations $n^\mathbb{Q}$, $f^\mathbb{Q}$, $p^\mathbb{R}$, and $F^\mathbb{R}$ as well.

5.2 Examples

The real line $\mathbb{R}$ a computable metric space $(\mathbb{R}, d_\mathbb{R}, \mathbb{Q}, \nu_\mathbb{Q})$ where $d_\mathbb{R}(x,y) = |x-y|$. Computability of a real number defined by Definition def2:comp.real coincides with that by Definition 2 in §2.1.

Similarly, a Euclidean space $\mathbb{R}^n$ is considered a computable metric space $(\mathbb{R}^n, d_{\mathbb{R}^n}, \mathbb{Q}^n, \nu_{\mathbb{Q}^n})$ with the Euclidean metric $d_{\mathbb{R}^n}$, the set of all rational points $\mathbb{Q}^n$ and the enumeration $\nu_{\mathbb{Q}^n}(\langle\!\langle i_1, \ldots, i_n \rangle\!\rangle) = (\nu_\mathbb{Q}(i_1), \ldots, \nu_\mathbb{Q}(i_n))$.

The function space $C[0,1]$, all continuous real-valued functions on $[0,1]$, with the max norm $\|f\| = \max_{x\in[0,1]}|f(x)|$ forms a computable metric space with the set of all polynomial functions with rational coefficients as the dense subset and an effective enumeration of the dense subset.

Chapter 6

Urysohn's universal metric space

Urysohn [16] proved that there exists a separable metric space U such that

1. for any separable metric space X, there exists an isometric embedding from X into U, and
2. for any $x_0, \dots, x_{s-1} \in U$ and any $\alpha_0, \dots, \alpha_{s-1} \in \mathbb{R}$, if

   $$(\forall i, j)\ |\alpha_i - \alpha_j| \leq d_U(x_i, x_j) \leq \alpha_i + \alpha_j,$$

   then there exists $y \in U$ such that

   $$(\forall i)\ d_U(x_i, y) = \alpha_i.$$

Urysohn also proved that such a space U is unique up to isometry. U is called *Urysohn's universal metric space*.

In this chapter, we investigate Urysohn's universal metric space from the point of view of computability.

6.1 Computability

6.1.1 Construction of U_0

We follow Urysohn's construction of U_0 (Chap. III in [16]) verifying computability of each step.

Let (Q_n) be an effective enumeration of all finite sequences of positive rational numbers such that $\operatorname{len} Q_n \leq n$.

We define $d_{U_0}(i,j)$ inductively on $\max\{i,j\}$. In the nth step of the induction, we first define $d_{U_0}(n,n)=0$. Next, we define $d_{U_0}(n,j)$ for $j<n$ as follows. Let $(r_0^{(n)},\dots,r_{s_n-1}^{(n)})=Q_n$. We test the condition

$$(\forall k,k'<s_n)\quad |r_k^{(n)}-r_{k'}^{(n)}|\le d_{U_0}(k,k')\le r_k^{(n)}+r_{k'}^{(n)}, \tag{$\triangle$}$$

and then define

$$d_{U_0}(n,j)=\begin{cases}\min\{d_{U_0}(j,k)+r_k^{(n)}:k<s_n\} & \text{if } (\triangle) \text{ holds,}\\ \max\{d_{U_0}(k,k'):k,k'<n\} & \text{otherwise.}\end{cases}$$

Finally, we define $d_{U_0}(j,n)=d_{U_0}(n,j)$ for $j<n$.

It is clear that there exists a recursive function $\rho_0:\mathbb{N}\times\mathbb{N}\to\mathbb{N}$ such that $d_{U_0}={\rho_0}^{\mathbb{Q}}$.

Urysohn proved that $(\mathbb{N},d_{U_0})$ forms a metric space (§10 in [16]). As [16] is written in French and there is no English translation so far as we know, we outline the proof here.

Positivity, identity, and symmetry are trivial. We prove the triangle inequality $d_{U_0}(i,j)+d_{U_0}(j,k)\ge d_{U_0}(i,k)$ by induction on $\max\{i,j,k\}$. It is sufficient to show that

$$(\forall i,j,k<n)\quad d_{U_0}(i,j)+d_{U_0}(j,k)\ge d_{U_0}(i,k)$$

implies

$$d_{U_0}(i,n)+d_{U_0}(n,j)\ge d_{U_0}(i,j),$$
$$d_{U_0}(n,i)+d_{U_0}(i,j)\ge d_{U_0}(n,i)$$

for $i,j<n$. In the case the condition $(\triangle)$ holds for n, there exist some $k_0,k_0'<s_n$ such that

$$d_{U_0}(n,i)=d_{U_0}(i,k_0)+r_{k_0}^{(n)},\qquad d_{U_0}(n,j)=d_{U_0}(j,k_0')+r_{k_0'}^{(n)},$$

Thus we have

$$\begin{aligned}d_{U_0}(i,n)+d_{U_0}(n,j)&=d_{U_0}(i,k_0)+d_{U_0}(j,k_0')+r_{k_0}^{(n)}+r_{k_0'}^{(n)}\\ &\ge d_{U_0}(i,k_0)+d_{U_0}(j,k_0')+d_{U_0}(k_0,k_0')\\ &\ge d_{U_0}(i,j).\end{aligned}$$

We also have

$$\begin{aligned}d_{U_0}(n,i)+d_{U_0}(i,j)&=d_{U_0}(i,k_0)+d_{U_0}(i,j)+r_{k_0}^{(n)}\\ &\ge d_{U_0}(k_0,j)+r_{k_0}^{(n)}\\ &\ge \min_{k<s_n}(d_{U_0}(k,j)+r_k^{(n)})\\ &=d_{U_0}(n,i).\end{aligned}$$

The case the condition $(\triangle)$ does not hold for n is trivial.

Theorem 5. *There exists a recursive function* $\gamma : (\mathbb{N} \times \mathbb{N})^* \to \mathbb{N}$ *such that for any* $m_0, \ldots, m_{s-1} \in \mathbb{N}$ *and any* $n_0, \ldots, n_{s-1} \in \mathbb{N}$, *if*

$$(\forall i, j < s) \quad |{n_i}^{\mathbb{Q}} - {n_j}^{\mathbb{Q}}| \leq d_{U_0}(m_i, m_j) \leq {n_i}^{\mathbb{Q}} + {n_j}^{\mathbb{Q}}, \tag{6.1}$$

then

$$(\forall i < s) \quad d_{U_0}(m_i, \gamma(m_0, n_0, \ldots, m_{s-1}, n_{s-1})) = {n_i}^{\mathbb{Q}}.$$

Proof. For $m_0, \ldots, m_{s-1} \in \mathbb{N}$ and $n_0, \ldots, n_{s-1} \in \mathbb{N}$ satisfying (6.1), we define $\gamma(m_0, n_0, \ldots, m_{s-1}, n_{s-1})$ as follows.

Let $s' = \max\{m_0, \ldots, m_{s-1}\} + 1$. Let n'_k be some appropriate natural number such that ${n'_k}^{\mathbb{Q}} = \min_{i<s}(d_{U_0}(k, m_i) + {n_i}^{\mathbb{Q}})$ for each $k < s'$. We have ${n'_{m_i}}^{\mathbb{Q}} = {n_i}^{\mathbb{Q}}$ since $d_{U_0}(m_i, m_{i'}) + {n_{i'}}^{\mathbb{Q}} \geq {n_i}^{\mathbb{Q}}$ for all $i' < s$ and $d_{U_0}(m_i, m_{i'}) + {n_{i'}}^{\mathbb{Q}} = {n_i}^{\mathbb{Q}}$ if $i' = i$. In other words,

$$\{(m_0, {n_0}^{\mathbb{Q}}), \ldots, (m_{s-1}, {n_{s-1}}^{\mathbb{Q}})\} \subseteq \{(0, {n_0}^{\mathbb{Q}}), \ldots, (s'-1, {n_{s'-1}}^{\mathbb{Q}})\}.$$

We have, for some $i < s$,

$$\begin{aligned} d(k, l) + {n'_k}^{\mathbb{Q}} &= d(k, l) + d(k, m_i) + {n_i}^{\mathbb{Q}} \\ &\geq d(l, m_i) + {n_i}^{\mathbb{Q}} \\ &\geq \min_{i<s}(d(l, m_i) + {n_i}^{\mathbb{Q}}) = {n'_l}^{\mathbb{Q}}. \end{aligned}$$

Thus ${n'_l}^{\mathbb{Q}} - {n'_k}^{\mathbb{Q}} \leq d(k, l)$. Similarly, we obtain ${n'_k}^{\mathbb{Q}} - {n'_l}^{\mathbb{Q}} \leq d(k, l)$. We also have, for some $i, j < s$,

$$\begin{aligned} {n'_k}^{\mathbb{Q}} + {n'_l}^{\mathbb{Q}} &= d_{U_0}(k, m_i) + d_{U_0}(l, m_j) + {n_i}^{\mathbb{Q}} + {n_j}^{\mathbb{Q}} \\ &\geq d_{U_0}(k, m_i) + d_{U_0}(l, m_j) + d_{U_0}(m_i, m_j) \\ &\geq d_{U_0}(k, l). \end{aligned}$$

Therefore,

$$(\forall k, l < s') \quad |{n'_k}^{\mathbb{Q}} - {n'_l}^{\mathbb{Q}}| \leq d_{U_0}(k, l) \leq {n'_k}^{\mathbb{Q}} + {n'_l}^{\mathbb{Q}}. \tag{6.2}$$

Define $\gamma : \mathbb{N} \to \mathbb{N}$ by

$$\gamma(m_0, n_0, \ldots, m_{s-1}, n_{s-1}) = \min\{N : Q_N = ({n'_0}^{\mathbb{Q}}, \ldots, {n'_{s'-1}}^{\mathbb{Q}}).$$

It is clear that γ is a recursive total function. From the definition of d_{U_0} as well as (6.2), we obtain

$$d_{U_0}(k, \gamma(m_0, n_0, \ldots, m_{s-1}, n_{s-1})) = {n'_k}^{\mathbb{Q}},$$

and particularly

$$d_{U_0}(m_i, \gamma(m_0, n_0, \ldots, m_{s-1}, n_{s-1})) = {n_i}^{\mathbb{Q}}.$$

□

6.1.2 Construction of U from U_0

Let (U, d_U) be a completion of $(\mathbb{N}, d_{U_0})$. We denote the inclusion by $\nu_{\mathbb{Q}}$. We denote by U_0 the image of the inclusion of $\mathbb{N}$ into U. Then (U, d_U, U_0, ν_{U_0}) is a computable metric space.

We state three propositions without proofs here. The proofs are tedious but straightforward. We use Propositions 5, 6, and 7 in the proof of Lemma 10 below. We also use Proposition 6 in the proof of Theorem 6 below.

Proposition 5. *There exists a computable function $\Theta_1 :\subseteq \mathbb{N}^\omega \times \mathbb{N}^\omega \to \mathbb{N}$ such that if $p, q \in \mathrm{dom}(\delta_{\mathbb{R}})$ and $p^{\mathbb{R}} < q^{\mathbb{R}}$, then $p^{\mathbb{R}} < {\Theta_1}^{\mathbb{Q}}(p, q) < q^{\mathbb{R}}$.*

Proposition 6. *There exists a computable function $\Theta_2 :\subseteq \mathbb{N} \times \mathbb{N} \times (\mathbb{N}^\omega)^* \to \mathbb{N}$ such that if $k^{\mathbb{Q}} < l^{\mathbb{Q}}$ and $p_0, \ldots, p_{s-1} \in \mathrm{dom}(\delta_{\mathbb{R}})$, then the following hold.*

- *$(k, l, p_0, \ldots, p_{s-1}) \in \mathrm{dom}(\Theta_2)$.*
- *If $\Theta_2(k, l, p_0, \ldots, p_{s-1}) \geq s$, then $(\forall i < s)\ {p_i}^{\mathbb{R}} > k^{\mathbb{Q}}$.*
- *If $\Theta_2(k, l, p_0, \ldots, p_{s-1}) = i_0 < s$, then ${p_{i_0}}^{\mathbb{R}} < l^{\mathbb{Q}}$.*

Proposition 7. *There exists a computable function $\Theta_3 :\subseteq \mathbb{N} \times (\mathbb{N}^\omega)^* \to \mathbb{N}^*$ such that if $k^{\mathbb{Q}} > 0$ and $p_0, \ldots, p_{s-1} \in \mathrm{dom}(\delta_{\mathbb{R}})$, then the following hold.*

- *$\Theta_3(k, p_0, \ldots, p_{s-1})$ is defined and is a permutation of $(0, \ldots, s-1)$.*
- *Let $(n_0, \ldots, n_{s-1}) = \Theta_3(k, p_0, \ldots, p_{s-1})$. Then, for any $i, j < s$, if $i < j$, then ${p_{n_i}}^{\mathbb{R}} + k^{\mathbb{Q}} > {p_{n_j}}^{\mathbb{R}}$.*

Lemma 10. *There exists a computable function $\Gamma_1 :\subseteq \mathbb{N} \times (\mathbb{N}^\omega \times \mathbb{N}^\omega)^* \to \mathbb{N}$ such that for any $k \in \mathbb{N}$, any $p_0, \ldots, p_{s-1} \in \mathrm{dom}(\delta_U)$, and any $q_0, \ldots, q_{s-1} \in \mathrm{dom}(\delta_{\mathbb{R}})$, if*

$$k^{\mathbb{Q}} > 0 \quad \textit{and} \quad (\forall i, j < s) \quad |{q_i}^{\mathbb{R}} - {q_j}^{\mathbb{R}}| \leq d_U({p_i}^U, {p_j}^U) \leq {q_i}^{\mathbb{R}} + {q_j}^{\mathbb{R}}, \quad (6.3)$$

then

$$(\forall i < s) \quad {q_i}^{\mathbb{R}} - k^{\mathbb{Q}} < d_U({p_i}^U, {\Gamma_1}^{U_0}(k, p_0, q_0, \ldots, p_{s-1}, q_{s-1})) < {q_i}^{\mathbb{R}} + k^{\mathbb{Q}}.$$

Proof. We show existence of a computable function Γ_1 by constructing a procedure that computes $\Gamma_1(k, p_0, q_0, \dots, p_{s-1}, q_{s-1})$ for a given number $k \in \mathbb{N}$, given sequences $p_0, \dots, p_{s-1} \in \mathrm{dom}(\delta_U)$, and given sequences $q_0, \dots, q_{s-1} \in \mathrm{dom}(\delta_{\mathbb{R}})$. Throughout this proof, we suppose (6.3) and write $k^{\mathbb{Q}}$ as ε, ${p_i}^U$ as x_i, and ${q_i}^{\mathbb{R}}$ as α_i.

Let $\varphi_1, \varphi_2 : \mathbb{N} \to \mathbb{N}$ be recursive functions such that ${\varphi_1}^{\mathbb{Q}}(k) = k^{\mathbb{Q}}/5$ and ${\varphi_2}^{\mathbb{Q}}(k) = 2k^{\mathbb{Q}}/5$. Define $s' \in \mathbb{N}$ and $\sigma, \sigma' :\subseteq \{0, \dots, s-1\} \to \{0, \dots, s-1\}$ by the following algorithm using the function Θ_2 in Proposition 6.

$\sigma(0) := 0;$
$s' := 1;$
for i **from** 1 **to** $s-1$ **do**
  $j := \Theta_2(\varphi_1(k), \varphi_2(k), \rho_U(p_{\sigma(0)}, p_i), \dots, \rho_U(p_{\sigma(s'-1)}, p_i));$
  if $j < s'$ **then**
    $\sigma'(i) := j;$
  else
    $\sigma(s') := i;$
    $s' := s' + 1;$
  fi
od

It is easy to verify that $1 \leq s' \leq s$, $\mathrm{dom}(\sigma') = \{0, \dots, s-1\} \setminus \mathrm{range}(\sigma)$, and $\mathrm{range}(\sigma') \subseteq \mathrm{dom}(\sigma)$. It is also easy to verify

$$(\forall i, j \in \mathrm{range}(\sigma), i \neq j)\ d_U(p_i, p_j) > \frac{1}{5}\varepsilon,$$
$$(\forall i \in \mathrm{dom}(\sigma'))\ d_U(p_i, p_{\sigma'(i)}) < \frac{2}{5}\varepsilon.$$

Let $\varepsilon' = \varepsilon/(15s' + 5)$. By using the function Θ_3 in Proposition 7, we can find a permutation $(t_0, \dots, t_{s'-1})$ of $(0, \dots, s'-1)$ such that for any $i, j < s'$, if $i < j$, then $\alpha_{\sigma(t_i)} + \varepsilon' > \alpha_{\sigma(t_j)}$. We have $(\forall i, j < s')\ d_U(x_{\sigma(t_i)}, x_{\sigma(t_j)}) > \varepsilon/5$. We write $x_{\sigma(t_i)}$ as x'_i and $\alpha_{\sigma(t_i)}$ as α'_i.

We can effectively find $m_i, n_i \in \mathbb{N}$, for each $i < s'$, such that

$$d_U(x'_i, m_i^{U_0}) < \varepsilon',$$
$$\alpha'_i + 3i\varepsilon' < n_i^{\mathbb{Q}} < \alpha'_i + (3i+1)\varepsilon'.$$

Finding each m_i is straightforward. In finding each n_i, we use the function Θ_1 in Proposition 5. We write $m_i^{U_0}$ as a_i and $n_i^{\mathbb{Q}}$ as β_i.

We will verify $|\beta_i - \beta_j| \le d_U(a_i, a_j) \le \beta_i + \beta_j$. Without loss of generality, we can assume $i < j$. We have

$$\begin{aligned} \beta_j - \beta_i &< \alpha'_j - \alpha'_i + (3j - 3i + 1)\varepsilon' \\ &< (3s' + 1)\varepsilon' - \varepsilon' - \varepsilon' \\ &< d(x'_i, x'_j) - d(x'_i, a_i) - d(x'_j, a_j) \\ &\le d(a_i, a_j). \end{aligned}$$

We also have

$$\begin{aligned} \beta_i - \beta_j &< \alpha'_i - \alpha'_j + (3i - 3j + 1)\varepsilon' \\ &< |\alpha'_i - \alpha'_j| - \varepsilon' - \varepsilon' \\ &< d(x'_i, x'_j) - d(x'_i, a_i) - d(x'_j, a_j) \\ &\le d(a_i, a_j). \end{aligned}$$

We also have

$$\begin{aligned} \beta_i + \beta_j &> \alpha'_i + \alpha'_j + (3i + 3j)\varepsilon' \\ &> \alpha'_i + \alpha'_j + \varepsilon' + \varepsilon' \\ &> d(x'_i, x'_j) + d(x'_i, a_i) + d(x'_j, a_j) \\ &\ge d(a_i, a_j). \end{aligned}$$

Let $m' = \gamma(m_0, n_0, \dots, m_{s'-1}, n_{s'-1})$ and $b = m'^{U_0}$. As an application of Theorem 5, we obtain $d_U(a_i, b) = \beta_i$. Thus we have

$$\begin{aligned} d_U(x'_i, b) &\le d_U(a_i, b) + d_U(x'_i, a_i) \\ &< \beta_i + \varepsilon' \\ &< \alpha'_i + (3i + 2)\varepsilon' \\ &< \alpha'_i + (3s' + 1)\varepsilon' \\ &< \alpha'_i + \frac{1}{5}\varepsilon. \end{aligned}$$

We also have

$$\begin{aligned} d_U(x'_i, b) &\ge d_U(a_i, b) - d_U(x'_i, a_i) \\ &> \beta_i - \varepsilon' \\ &> \alpha'_i + (3i - 1)\varepsilon' \\ &> \alpha'_i - \varepsilon' \\ &> \alpha'_i - \frac{1}{5}\varepsilon. \end{aligned}$$

Define $\Gamma_1(k, p_0, q_0, \ldots, p_{s-1}, q_{s-1}) = m'$. We are now ready to complete the proof. In the case $i \in \mathrm{range}(\sigma)$, we have already done. In the case $i \in \mathrm{dom}(\sigma')$, let $j = \sigma'(i)$. We have

$$d_U(x_i, x_j) < \frac{2}{5}\varepsilon.$$

We have a triangle inequality

$$d_U(x_j, b) - d_U(x_i, x_j) \leq d_U(x_i, b) \leq d_U(x_j, b) + d_U(x_i, x_j).$$

Since $j \in \mathrm{dom}(\sigma)$, we have

$$\alpha_j - \frac{1}{5}\varepsilon < d_U(x_j, b) < \alpha_j + \frac{1}{5}\varepsilon.$$

From the condition (6.3), we obtain

$$\alpha_i - d_U(x_i, x_j) \leq \alpha_j \leq \alpha_i + d_U(x_i, x_j).$$

From these four inequalities, we obtain $\alpha_i - \varepsilon < d_U(x_i, b) < \alpha_i + \varepsilon$. $\square$

Theorem 6. *There exists a computable function $\Gamma :\subseteq (\mathbb{N}^\omega \times \mathbb{N}^\omega)^* \to \mathbb{N}^\omega$ such that for any $p_0, \ldots, p_{s-1} \in \mathrm{dom}(\delta_U)$ and any $q_0, \ldots, q_{s-1} \in \mathrm{dom}(\delta_\mathbb{R})$, if*

$$(\forall i, j < s) \quad |{q_i}^\mathbb{R} - {q_j}^\mathbb{R}| \leq d_U({p_i}^U, {p_j}^U) \leq {q_i}^\mathbb{R} + {q_j}^\mathbb{R}, \tag{6.4}$$

then

$$(\forall i < s) \quad d_U({p_i}^U, \Gamma^U(p_0, q_0, \ldots, p_{s-1}, q_{s-1})) = {q_i}^\mathbb{R}.$$

Proof. We show existence of a computable function Γ by constructing a procedure that computes $\Gamma(p_0, q_0, \ldots, p_{s-1}, q_{s-1})$ for given sequences $p_0, \ldots, p_{s-1} \in \mathrm{dom}(\delta_U)$ and given sequences $q_0, \ldots, q_{s-1} \in \mathrm{dom}(\delta_\mathbb{R})$. Throughout this proof, we suppose (6.4) and write ${p_i}^U$ as x_i and ${q_i}^\mathbb{R}$ as α_i.

Define $p' \in \mathrm{dom}(\delta_U)$ by the following algorithm with the function Θ_1 in Proposition 5, the function Θ_2 in Proposition 6, and the function Γ_1 in Lemma 10.

> Let S_0 be a subset of $\{0, \ldots, s-1\}$ such that
> $\quad(\forall j \notin S_0)(\exists j' \in S_0)\ d_U({p_j}^U, {p_{j'}}^U) < 1,$
> $\quad(\forall j.j', \in S_0)\ d_U({p_j}^U, {p_{j'}}^U) > 0;$
> By using Θ_1, find a natural number k_0 such that
> $\quad 0 < {k_0}^\mathbb{Q} < \min(\{d_U({p_j}^U, {p_{j'}}^U) : j, j' \in S_0\} \cup \{1\});$
> $p'[0] := \Gamma_1(k_0, p_{j_0}, q_{j_0}, \ldots, p_{j_{s'-1}}, q_{j_{s'-1}})$ where $\{j_0, \ldots, j_{s'-1}\} = S_0$;

for i **from** 1 **to** ∞ **do**
  Let S_i be a subset of $\{0, \ldots, s-1\}$ such that
    $S_i \supseteq S_{i-1}$,
    $(\forall j \notin S_{i+1})(\exists j' \in S_{i+1})\ d_U({p_j}^U, {p_{j'}}^U) < 1/2^i$,
    $(\forall j.j', \in S_{i+1})\ d_U({p_j}^U, {p_{j'}}^U) > 0$;
  By using Θ_1, find a natural number k_i such that
    $0 < {k_i}^{\mathbb{Q}} < \min(\{d_U({p_j}^U, {p_{j'}}^U) : j, j' \in S_i\} \cup \{1/2^i\})$;
  $p'[i] := \Gamma_1(k_i, p_{j_0}, q_{j_0}, \ldots, p_{j_{s'-1}}, q_{j_{s'-1}}, p'[i-1], (k_i)_{i' \in \mathbb{N}})$ where
    $\{j_0, \ldots, j_{s'-1}\} = S_i$;
od

Note that $(k_i)_{i' \in \mathbb{N}}$ denotes a sequence repeating k_i forever, namely a sequence such that the i'-th element is k_i whatever the index i' is

It follows immediately from this algorithm that $d_U(p[i]^{U_0}, p[i+1]^{U_0}) < 1/2^{i+1}$. Hence if $i < j$, then $d_U(p[i]^{U_0}, p[j]^{U_0}) < 1/2^i$. Hence $p \in \delta_U$. Thus let $y = p^U$.

If $j \in \bigcup_{i=0}^{\infty} S_i$, then

$$(\exists i_0)(\forall i \geq i_0) \quad {q^j}^{\mathbb{R}} - \frac{1}{2^i} < d_U({p_j}^U, p'[i]^{U_0}) < {q^j}^{\mathbb{R}} + \frac{1}{2^i}.$$

Hence $d_U(x_j, y) = \alpha_j$. If $j \notin \bigcup_{i=0}^{\infty} S_i$, then $x_j = x_{j'}$ and $\alpha_j = \alpha_{j'}$ for some $j' \in \bigcup_{i=0}^{\infty} S_i$. Hence $d_U(x_j, y) = d_U(x_{j'}, y) = \alpha_{j'} = \alpha_j$.

Defining $\Gamma(p_0, q_0, \ldots, p_{s-1}, q_{s-1}) = p'$, we complete the proof. $\square$

6.2 Computable isometric embedding

6.2.1 Embedding of a computable metric space into U

Urysohn proved that for any separable metric space X, there exists an isometric embedding from X to U. We show that for any computable metric space X, there exists an isometric computable embedding from X to U.

Define $\rho_A : \mathbb{N} \times \mathbb{N} \to \mathbb{N}^\omega$ by $\rho_A(m, n) = \rho_X((m)_{i \in \mathbb{N}}, (n)_{i \in \mathbb{N}})$. Here, $(n)_{i \in \mathbb{N}}$ denotes a sequence such that the i-th element is n whatever the index i is. Note that ${\rho_A}^{\mathbb{R}}(i, j) = d_X(i^A, j^A)$.

Define $(u_n) \in (\mathbb{N}^\omega)^\omega$ by

$$\begin{cases} u_0 = (0)_{i \in \mathbb{N}}, \\ u_n = \Gamma(u_0, \rho_A(0, n), \ldots, u_{n-1}, \rho_A(n-1, n)) & \text{if } n \geq 1. \end{cases}$$

Then, $d_U({u_m}^U, {u_n}^U) = d_X(m^A, n^A)$. It is trivial that (u_n) is a computable sequence of sequences, i.e., $(u_n[i])_{n, i \in \mathbb{N}}$ forms a recursive double sequence of natural numbers.

Define $I : \mathbb{N}^\omega \to \mathbb{N}^\omega$ by $I(p) = (u_{p[i+2]}[i+1])_{i\in\mathbb{N}}$. It is clear that I is a computable function.

Suppose $p \in \mathrm{dom}(\delta_X)$. If $i < j$, then

$$\begin{aligned}
&d_U(I(p)[i]^{U_0}, I(p)[j]^{U_0}) \\
&= d_U(u_{p[i+2]}[i+1]^{U_0}, u_{p[j+2]}[j+1]^{U_0}) \\
&\leq d_U(u_{p[i+2]}[i+1]^{U_0}, u_{p[i+2]}{}^U) + d_U(u_{p[i+2]}{}^U, u_{p[j+2]}{}^U) \\
&\qquad + d_U(u_{p[j+2]}[j+1]^{U_0}, u_{p[j+2]}{}^U) \\
&\leq \frac{1}{2^{i+1}} + \frac{1}{2^{i+2}} + \frac{1}{2^{j+1}} \\
&\leq \frac{1}{2^{i+1}} + \frac{1}{2^{i+2}} + \frac{1}{2^{i+2}} = \frac{1}{2^i}.
\end{aligned}$$

Hence $p \in \mathrm{dom}(\delta_X)$ implies $I(p) \in \mathrm{dom}(\delta_U)$.

Suppose $p, q \in \mathrm{dom}(\delta_X)$. We have

$$\begin{aligned}
&d_U(u_{p[i+2]}[i+1]^{U_0}, u_{q[j+2]}[j+1]^{U_0}) \\
&\leq d_U(u_{p[i+2]}{}^U, u_{q[j+2]}{}^U) \\
&\qquad + d_U(u_{p[i+2]}[i+1]^{U_0}, u_{p[i+2]}{}^U) + d_U(u_{q[j+2]}[j+1]^{U_0}, u_{q[j+2]}{}^U) \\
&\leq d_X(p[i+2]^A, q[j+2]^A) + \frac{1}{2^{i+1}} + \frac{1}{2^{j+1}} \\
&\to d_X(p^X, q^X) \qquad \text{as } i, j \to \infty.
\end{aligned}$$

We also have

$$\begin{aligned}
&d_U(u_{p[i+2]}[i+1]^{U_0}, u_{q[j+2]}[j+1]^{U_0}) \\
&\geq d_U(u_{p[i+2]}{}^U, u_{q[j+2]}{}^U) \\
&\qquad - d_U(u_{p[i+2]}[i+1]^{U_0}, u_{p[i+2]}{}^U) - d_U(u_{q[j+2]}[j+1]^{U_0}, u_{q[j+2]}{}^U) \\
&\geq d_X(p[i+2]^A, q[j+2]^A) - \frac{1}{2^{i+1}} - \frac{1}{2^{j+1}} \\
&\to d_X(p^X, q^X) \qquad \text{as } i, j \to \infty.
\end{aligned}$$

Thus

$$\lim_{i,j\to\infty} d_U(u_{p[i+2]}[i+1]^{U_0}, u_{q[j+2]}[j+1]^{U_0}) = d_X(p^X, q^X).$$

Hence, $p, q \in \mathrm{dom}(\delta_X)$ implies $d_U(I(p)^U, I(q)^U) = d_X(p^X, q^X)$.

Defining $\iota : X \to U$ by $\iota(p^X) = I^U(p)$, we obtain the following theorem.

Theorem 7. *For any computable metric space (X, d_X, A, ν_A), there exists a computable, isometric function $\iota : X \to U$.*

6.3 Computable uniqueness

6.3.1 Uniqueness of U_0

We follow Joiner's "back and forth" method (Theorem 2 in [9]) verifying its effectiveness.

Lemma 11. *Let* $\rho' : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ *and* $\gamma' : (\mathbb{N}\times\mathbb{N})^* \to \mathbb{N}$ *be recursive functions. Suppose* ${\rho'}^{\mathbb{Q}}$ *forms a metric on* $\mathbb{N}$. *Suppose for any* $m_0, \ldots, m_{s-1} \in \mathbb{N}$ *and any* $n_0, \ldots, n_{s-1} \in \mathbb{N}$, *if*

$$(\forall i, j < s) \quad |{n_i}^{\mathbb{Q}} - {n_j}^{\mathbb{Q}}| \le {\rho'}^{\mathbb{Q}}(m_i, m_j) \le {n_i}^{\mathbb{Q}} + {n_j}^{\mathbb{Q}},$$

then

$$(\forall i < s) \quad {\rho'}^{\mathbb{Q}}(m_i, {\gamma'}^{\mathbb{Q}}(m_0, n_0, \ldots, m_{s-1}, n_{s-1})) = {n_i}^{\mathbb{Q}}.$$

Then, there exists a recursive, bijective function $\varphi : \mathbb{N} \to \mathbb{N}$ *such that for any* $m, m' \in \mathbb{N}$, *it holds that* $d_{U_0}(m, m') = {\rho'}^{\mathbb{Q}}(\varphi(m), \varphi(m'))$.

Proof. Let $\rho_0 : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a recursive function such that $d_{U_0} = {\rho_0}^{\mathbb{Q}}$. We define (λ_i) and (λ'_i) by the following algorithm.

$\lambda_0 := 0$; $\lambda'_0 := 0$;
for i **from** 1 **to** ∞ **do**
  if i is odd **then**
    $\lambda'_i := \min(\mathbb{N} \setminus \{\lambda'_0, \ldots, \lambda'_{i-1}\})$;
    $\lambda_i := \gamma(\lambda_0, \rho'(\lambda'_0, \lambda'_i), \ldots, \lambda_{i-1}, \rho'(\lambda'_{i-1}, \lambda'_i))$;
  else
    $\lambda_i := \min(\mathbb{N} \setminus \{\lambda_0, \ldots, \lambda_{i-1}\})$;
    $\lambda'_i := \gamma'(\lambda'_0, \rho_0(\lambda_0, \lambda_i), \ldots, \lambda'_{i-1}, \rho_0(\lambda_{i-1}, \lambda_i))$;
  fi
od

And then we define φ by $\varphi(\lambda_i) = \lambda'_i$.

This is not precisely an algorithm because it never terminates. We can however modify this to be an algorithm by inserting some commands to stop just after $\varphi(n)$ is determined for a given n.

It is clear that φ is a recursive function. It is straightforward to show that φ is an isometric bijection between $(\mathbb{N}, d_{U_0})$ and $(\mathbb{N}, {\rho'}^{\mathbb{Q}})$. □

6.3.2 Construction of U'_0 in U'

Lemma 12. *Let $(U', d_{U'}, A', \nu_{A'})$ be a computable metric space. Let $\Gamma' :\subseteq (\mathbb{N}^\omega \times \mathbb{N}^\omega)^* \to \mathbb{N}^\omega$ be a computable function such that for any $p_0, \ldots, p_{s-1} \in \mathrm{dom}(\delta_{U'})$ and any $q_0, \ldots, q_{s-1} \in \mathrm{dom}(\delta_{\mathbb{R}})$, if*

$$(\forall i, j < s) \quad |{q_i}^{\mathbb{R}} - {q_j}^{\mathbb{R}}| \leq d_{U'}({p_i}^{U'}, {p_j}^{U'}) \leq {q_i}^{\mathbb{R}} + {q_j}^{\mathbb{R}},$$

then

$$(\forall i < s) \quad d_{U'}({p_i}^{U'}, {\Gamma'}^{U'}(p_0, q_0, \ldots, p_{s-1}, q_{s-1})) = {q_i}^{\mathbb{R}}.$$

Then, there exist a dense subset $U'_0 \subset U'$ with an enumeration $\nu_{U'_0} : \mathbb{N} \to U'_0$, a computable function $\Lambda' : \mathbb{N} \to \mathbb{N}^\omega$, and recursive functions $\rho' : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ and $\gamma' : (\mathbb{N} \times \mathbb{N})^* \to \mathbb{N}$ such that the following three conditions hold.

1. *It holds that $\nu_{U'_0} = \delta_{U'} \circ \Lambda'$.*
2. *For any $m, m' \in \mathbb{N}$, it holds that ${\rho'}^{\mathbb{Q}}(m, m') = d_{U'}(\nu_{U'_0}(m), \nu_{U'_0}(m'))$.*
3. *For any $m_0, \ldots, m_{s-1} \in \mathbb{N}$ and any $n_0, \ldots, n_{s-1} \in \mathbb{N}$, if*

   $$(\forall i, j < s) \quad |{n_i}^{\mathbb{Q}} - {n_j}^{\mathbb{Q}}| \leq d_{U'}(\nu_{U'_0}(m_i), \nu_{U'_0}(m_j)) \leq {n_i}^{\mathbb{Q}} + {n_j}^{\mathbb{Q}},$$

   then

   $$(\forall i < s) \quad d_{U'}(\nu_{U'_0}(m_i), \gamma'(\nu_{U'_0}(m_0), n_0, \ldots, \nu_{U'_0}(m_{s-1}), n_{s-1})) = {n_i}^{\mathbb{Q}}.$$

Sketch of the proof. We use the symbols Q_n and r_i^n defined in §6.1.1. Denote the value of the standard effective pairing function of m and n by $\langle m, n \rangle$. We assume $m \leq \langle m, n \rangle$.

We define ρ' and Λ' inductively as follows.

We define $\rho'(0, 0) = \ulcorner 0 \urcorner$ where $\ulcorner 0 \urcorner$ denotes the natural number such that $\ulcorner 0 \urcorner^{\mathbb{Q}} = 0$. We define $\Lambda'(0) = (0)_{i \in \mathbb{N}}$.

In the case $m = 2n + 1$, we first define $\rho'(m, m) = \ulcorner 0 \urcorner$. Nest, test the condition

$$(\forall k, k' < s_n) \quad |r_k^{(n)} - r_{k'}^{(n)}| \leq {\rho'}^{\mathbb{Q}}(k, k') \leq r_k^{(n)} + r_{k'}^{(n)} \qquad (\triangle^*)$$

and then let

$$\beta_j^{(m)} = \begin{cases} \min\{{\rho'}^{\mathbb{Q}}(j, k) + r_k^{(n)} : k < s_n\} & \text{if } (\triangle^*) \text{ holds,} \\ \max\{{\rho'}^{\mathbb{Q}}(k, k') : k, k' < m\} & \text{otherwise.} \end{cases}$$

for $j < m$. Using $\beta_j^{(m)}$, we define $\rho'(m,j)$ and $\rho'(j,m)$ for each j by ${\rho'}^{\mathbb{Q}}(m,j) = {\rho'}^{\mathbb{Q}}(j,m) = \beta_j^{(n)}$. Finally, we define

$$\Lambda'(m) = \Gamma(\Lambda'(0), \rho'(m,0), \ldots, \Lambda'(m), \rho'(m, m-1)).$$

In the case $m = 2\langle n,k\rangle + 2$, we first define $\rho'(m,m) = \ulcorner 0 \urcorner$. Next, find rational numbers $\beta_j^{(m)}$ for $j < m$, and a computable sequence $q''_m \in \delta_{\mathbb{R}}$ such that

$$\begin{gathered}
(\forall j, j' < m) \quad |\beta_j^{(m)} - \beta_{j'}^{(m)}| \leq {\rho'}^{\mathbb{Q}}(j,j') \leq \beta_j^{(m)} + \beta_{j'}^{(m)}, \\
(\forall j < m) \quad |\beta_j^{(m)} - {q''_m}^{\mathbb{R}}| \leq d_U(\nu_{U'_0}(j), \nu_{A'}(n)) \leq \beta_j^{(m)} + {q''_m}^{\mathbb{R}}, \\
{q''_m}^{\mathbb{R}} \leq \frac{1}{2^k}.
\end{gathered}$$

To find them, we use a trick similar to that used in the proof of Theorem 6. Using $\beta_j^{(m)}$, we define $\rho'(m,j)$ and $\rho'(j,m)$ for each j by ${\rho'}^{\mathbb{Q}}(m,j) = {\rho'}^{\mathbb{Q}}(j,m) = \beta_j^{(n)}$. Finally, we define

$$\Lambda'(m) = \Gamma(\Lambda'(0), \rho'(m,0), \ldots, \Lambda'(m-1), \rho'(m, m-1), (n)_{i\in\mathbb{N}}, q''_m).$$

From the case $m = 2n+1$, we obtain that the condition 3 holds. From the case $m = 2\langle n,k\rangle + 2$, we obtain that U'_0 is dense in U'.

We construct γ' similarly to the construction of γ in the proof of Theorem 5. □

6.3.3 Uniqueness of U

Theorem 8. *Let $(U', d_{U'}, A', \nu_{A'})$ be a computable metric space. Let $\Gamma' :\subseteq (\mathbb{N}^\omega \times \mathbb{N}^\omega)^* \to \mathbb{N}^\omega$ be a computable function such that for any $p_0, \ldots, p_{s-1} \in \mathrm{dom}(\delta_{U'})$ and any $q_0, \ldots, q_{s-1} \in \mathrm{dom}(\delta_{\mathbb{R}})$, if*

$$(\forall i, j < s) \quad |{q_i}^{\mathbb{R}} - {q_j}^{\mathbb{R}}| \leq d_{U'}({p_i}^{U'}, {p_j}^{U'}) \leq {q_i}^{\mathbb{R}} + {q_j}^{\mathbb{R}},$$

then

$$(\forall i < s) \quad d_{U'}({p_i}^{U'}, {\Gamma'}^{U'}(p_0, q_0, \ldots, p_{s-1}, q_{s-1})) = {q_i}^{\mathbb{R}}.$$

Then, there exists a computable, isometric, bijective function $\eta : U' \to U$.

Proof. It follows immediately from Lemma 11 and Lemma 12. □

Chapter 7

Conclusion

Some of the theorems in functional analysis or in general topology hold with some topological concepts replaced by corresponding computational concepts.

We have shown that counterparts in computable analysis of some fundamental theorems, the contraction theorem, Dini's theorem, and existence and uniqueness of Urysohn's universal metric space, hold. Furthermore, we have shown that not only these theorems are straightforward effectivization of the original theorems but also their proofs are straightforward effectivization of the original proofs.

In the cases of the contraction theorem and Dini's theorem, the proof of the computable counterpart shown in this paper is considered the original theorem for the non-computable theorem with verification of each step of the proof preserving computability. In the case of Urysohn's universal metric space, we need some modification in addition to verification of each step of the proof preserving computability. The modification, however, does not change the structure of the proof so much.

This straightforward effectivization will be useful in effectivication of applications of these theorems.

Acknowledgment

The author thanks to Dr. Masahiko Sato, Dr. Akihiro Yamamoto, and Dr. Taiichi Yuasa of Kyoto University for their helpful comments.

List of articles by the author

- H. Kamo, K. Kawamura: Computability of self-similar sets, Math. Log. Quart. **45** (1999) 23–30.
- Kamo, Hiroyasu; Kawamura, Kiko; Takeuti, Izumi: Computational complexity of fractal sets. Real Analysis Exchange **26** (2000/01), no. 2, 773–793.
- Kamo, Hiroyasu: Effective contraction theorem and its application. Computability and complexity in analysis (Swansea, 2000), 88–100, Lecture Notes in Comput. Sci., **2064**, Springer, Berlin, 2001.
- Kamo, Hiroyasu: Effective Dini's Theorem on Effectively Compact Metric Spaces, Electronic Notes in Theoretical Computer Science **120** (2005) 73–82.
- Hiroyasu Kamo. Computability and computable uniqueness of Urysohn's universal metric space. In Tanja Grubba, Peter Hertling, Hideki Tsuiki, and Klaus Weihrauch, editors, *Computability and Complexity in Analysis*, volume 326 of *Informatik Berichte*, pages 149–159. FernUniversität in Hagen, July 2005. Proceedings, Second International Conference, CCA 2005, Kyoto, Japan, August 25–29, 2005.

Bibliography

[1] C. Bandt: Self-Similar Sets 3. Constructions with Sofic systems, Monatshefte für Mathematik, **108** (1989) 89–102.

[2] V. Brattka, G. Presser: Computability on subsets of metric spaces, Theoret. Comput. Sci. **305** (2003) 43–76.

[3] V. Brattka, K. Weihrauch: Computability on subsets of Euclidean space I: Closed and compact subsets, Theoret. Comput. Sci. **219** (1999) 65–93.

[4] A. Grzegorczyk: Computable functional, Fund. Math. **42** (1955) 168–202.

[5] A. Grzegorczyk: On the definition of computable real functions, Fund. Math. **44** (1957) 61–71.

[6] M. Hata: On the structure of self-similar sets, Japan J. appl. Math., **2** (1985) 381–414.

[7] M.R. Holmes: The universal separable metric space of Urysohn and isometric embeddings thereof in Banach spaces, Fund. Math. **140** (1992) 199–223.

[8] J. E. Hutchinson: Fractals and self-similarity, Indiana Univ. Math. J., **30** (1981) 713–747.

[9] C. Joiner: On Urysohn's universal separable metric space, Fund. Math. **73** (1971) 51–58.

[10] H. Kamo, K. Kawamura: Computability of self-similar sets, Math. Log. Quart. **45** (1999) 23–30.

[11] Ch. Kreitz and K. Weihrauch: Theory of representations, Theoret. Comput. Sci. **38** (1985) 35–53.

[12] Subiman Kundu, A.B. Raha: The Stone-Weierstrass theorem and Dini's theorem: An insight, Houston J. of Math. **27** (2001) 887–895.

[13] T. Mori, Y. Tsujii, M. Yasugi: Computability structures on metric spaces, in: D. S. Bridges et al (Eds.), Combinatorics, Complexity and Logic, Proc. DMTCS '96, Springer, Berlin, (1996) 351–362.

[14] M. B. Pour-El, J. I. Richards: *Computability in Analysis and Physics*, Springer-Verlag, Berlin, Heidelberg, 1989.

[15] H. G. Rice: Recursive real numbers, Proc. Am. Math. Soc., **5** (1954) 784–791.

[16] P. Urysohn: Sur un espace métrique universel, Bull. Sci. Math. **51** (1927), 43–64, 74–90.

[17] K. Weihrauch: *Computability*, Springer-Verlag, Berlin, 1987.

[18] K. Weihrauch: Computability on computable metric spaces, Theoret. Comput. Sci. **113** (1993) 191–210.

[19] K. Weihrauch: *Computable Analysis*, Springer-Verlag, Berlin, 2000.

[20] M. Yasugi, T. Mori, Y. Tsujii: Effective properties of sets and functions in metric spaces with computability structure, Theoret. Comput. Sci. **219** (1999) 467–486.